Quantum Computing

Mathias Niepert

Homework 1

Exercise 11.1

What is the entropy associated with the toss of a fair coin?

Probabilistic experiment with two possible outcomes 0 and 1
Ω = {head, tail} Random variable X: $\Omega \rightarrow \{0, 1\}$. $P(X = 0) = p_0 = 1/2$ and $P(X = 1) = p_1 = 1/2$

$\Rightarrow$ H(X)$\equiv$ -$\sum_{j=0}^{1} p_i \log p_i$

= - (1/2log(1/2) + 1/2log(1/2)) = 1

What is the entropy associated with the toss of a fair die?

Probabilistic experiment with six possible outcomes $\Omega = \{1, .., 6\}$

Random variable X. $P(X = i) = p_i = 1/6$ $\forall$ i $\in \Omega$

$\Rightarrow$ H(X)$\equiv$ -1/6$\sum_{j=1}^{6}$ log(1/6) = 2.585

How would the entropy behave, if the coin or die were unfair?

In both cases the entropy would decrease, because at least one of the possible outcomes would appear with a higher probability then at least one of the others $\Rightarrow$ The system contains "less uncertainty"/the entropy of the system is smaller.

Exercise 11.2

The unlikely an event E is, the more information is gained, when it comes out. With an event with probability 1 no information is gained. So it seems appropriate to let the function I a function which takes the probability of the event as input and with: I(p)$\rightarrow$0 when p$\rightarrow$1 and I(p)$\rightarrow \pm \infty$ when p$\rightarrow$0. This is satisfied for I = $k$log(k).

A smooth function of probability f must meet the following properties:

1. $\int_D$f(x) = 1 where D is the domain (here: D = (0, 1] $\subset R$).
2. f(x) $\geq$ 0 $\forall$ x $\in D$.

This could be both satisfied for I = $k$log(p) and a certain k with $k < 0$.

It is: $k$log(pq) = k (log(p) + log(q)) = $k$log(p) + $k$log(q)

Exercise 11.6

(log(1-p))' = $\frac{-1}{(1-p)ln(2)}$
($p$log(p))' = log(p) + $\frac{1}{ln(2)}$
($p$log(1-p))' = - $\frac{p}{(1-p)ln(2)}$ + log(1-p)

$\Rightarrow$ $H_{bin}(p)$' = (-$p$log(p) - (1-p)log(1-p))' = log(1-p) - log(p)

log(1-p) - log(p) = 0 $\Rightarrow$
log($\frac{1-p}{p}$) = 0 $\Rightarrow$ p=1/2

$H_{bin}(p)$" = - ($\frac{1}{(1-p)ln(2)}$ + $\frac{1}{pln(2)}$) which is always < 0 for 0 < p < 1

$\Rightarrow$ Only one maximum for p=1/2

Show: States of a qubit is 1-1 correspondence with Bloch Sphere

State of a qubit: $|\psi\rangle = \alpha\ |0\rangle + \beta|1\rangle$
Probability for state $|0\rangle$ when measured: $|\alpha|^2$
Probability for state $|1\rangle$ when measured: $|\beta|^2$
$\rightarrow |\alpha|^2 + |\beta|^2 = 1$
Thus we can write:

$$|\psi\rangle = e^{i\varphi_1} cos(\frac{\theta}{2})|0\rangle + e^{i\varphi_2} sin(\frac{\theta}{2})|1\rangle =$$
$$e^{i\varphi_1}(cos(\frac{\theta}{2})|0\rangle + e^{i\varphi} sin(\frac{\theta}{2})|1\rangle \text{ ; with } \varphi_2 = \varphi_1 + \varphi$$

We can ignore the factor of e^{φ_1} because it has no observable physical effect.

$$\Rightarrow |\psi\rangle = cos(\tfrac{\theta}{2})|0\rangle + e^{i\varphi}sin(\tfrac{\theta}{2})|1\rangle =$$
$$cos(\tfrac{\theta}{2})|0\rangle + cos(\varphi)sin(\tfrac{\theta}{2})|1\rangle + sin(\varphi)sin(\tfrac{\theta}{2}) \text{ i } |1\rangle$$

This **defines** the (surface of the) Bloch Sphere, as we have three orthonormal base vectors ($|0\rangle$, $|1\rangle$, $i|1\rangle$) and, depending on the angles (θ, φ), the possible coordinates ($cos(\frac{\theta}{2})$, $cos(\varphi)sin(\frac{\theta}{2})$, $sin(\varphi)sin(\frac{\theta}{2})$). (By ignoring the factor $e^{i\varphi_1}$, we "removed" one dimension from our 4-dimensional C^2 space and the $|\alpha|^2 + |\beta|^2 = 1$ condition normed the length of all possible vectors to 1).

R^3, Bloch Sphere: All unit vectors in the z,x plane: $\begin{pmatrix} sin(\frac{\theta}{2}) \\ 0 \\ cos(\frac{\theta}{2}) \end{pmatrix}$

All possible rotations of these vectors round the z-axis:
$(0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi)$

$$\begin{pmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} sin(\frac{\theta}{2}) \\ 0 \\ cos(\frac{\theta}{2}) \end{pmatrix} = \begin{pmatrix} cos\varphi sin(\frac{\theta}{2}) \\ sin\varphi sin(\frac{\theta}{2}) \\ cos(\frac{\theta}{2}) \end{pmatrix}$$

Every possible value combination of the pair (θ, φ) defines exactly one (of all) possible states of a qubit and exactly one corresponding point on the Bloch Sphere. ($0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$)

$\Rightarrow$ There is a bijection between the states of a qubit and the points on the Bloch Sphere.

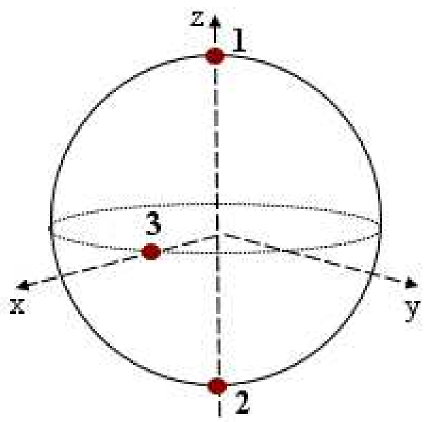

Figure 1: Bloch Sphere

1: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

But there are infinite many states with $|\beta| = \frac{1}{\sqrt{2}}$ and $\alpha = \frac{1}{\sqrt{2}}$