Quantum Computation

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Homework 10

Exercise 5.1

A linear transformation (operator) T is unitary, if $T \circ T^{\dagger} = I$

Matrix representation of F_n (n rows, n columns, $\omega = e^{2\pi i/n}$):

$$\frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & \omega & \omega^2 & ... & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & ... & \omega^{2(n-1)} \\ ... & ... & ... & ... & ... \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & ... & \omega^{(n-1)(n-1)} \end{pmatrix}$$

Matrix representation of $F_n^{\dagger}$ (n rows, n columns, $\omega = e^{-2\pi i/n}$):

$$\frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & \omega & \omega^2 & ... & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & ... & \omega^{2(n-1)} \\ ... & ... & ... & ... & ... \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & ... & \omega^{(n-1)(n-1)} \end{pmatrix}$$

The $j-th$ row of $F_n^{\dagger}$ is $\frac{1}{\sqrt{n}}(1, w^{-j}, w^{-2j}, ..., w^{(n-1)j})$

while the k-th column of F_n is $\frac{1}{\sqrt{n}}(1, w^{k}, w^{2k}, ..., w^{(n-1)k})$

The entry in the (j, k)-th place of the product $F_n F_n^{\dagger}$ is just $\frac{1}{n}(1 + \omega^{(j-k)} + \omega^{2(j-k)} + ... + \omega^{(n-1)(j-k)})$

Distinguish two cases: If j = k, this is just (1 + 1 + ... + 1)/n = 1, while if j≠k, it is $\frac{1}{n}(1 + \omega + \omega^2 + ... + \omega^{n-1})$ which is zero.

Proof: $\omega^{n(j-k)} = 1$, and $\omega^{(j-k)} \neq 1$.

$1 + \omega^{(j-k)} + \omega^{2(j-k)} + ... + \omega^{(n-1)(j-k)} = 0$ because $0 = \omega^{(j-k)n} - 1 = (\omega^{(j-k)} - 1)(1 + \omega^{(j-k)} + \omega^{2(j-k)} + ... + \omega^{(j-k)(n-1)})$ with $(\omega^{(j-k)} - 1) \neq 0$.

Exercise 5.3

$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi ijk/N}$

$N = 2^n$. Thus we have to evaluate the above formula 2^n times and every evaluation of the formula needs $N = 2^n$ steps. $\Rightarrow O(2^n \cdot 2^n) = O(2^{2n})$.

Formula 5.4 has to be evaluated 2^n times for every possible combinations of $j_1, ..., j_n (= 2^n)$. $\in O(2^n)$

It is possible to first evaluate $e^{2\pi i 0.j_1 j_2 \dots j_n} = e^{2\pi i(j_1/2 + j_2/2^2 + \dots + j_n/2^n)}$ in O(n) steps. Then one can multiply successively $e^{-2\pi i(j_k/2^k)}$ for $k = 1...n-1$ to get the coefficients for formula 5.4

$(e^{2\pi i 0.j_1 j_2 \dots j_n} \cdot e^{-2\pi i(j_1/2)}) = e^{2\pi i 0.j_2 \dots j_n}$... This is also possible in O(n).

In the end multiply the coefficients to the vector $|1\rangle$, add $|0\rangle$ and multiply the n vectors (size 2). This is possible in $\sim 5n$ steps $\in O(n)$.

Thus we need overall $O(n2^n)$ steps.

Exercise 5.5

See attached sheet.

Exercise B

$|\psi\rangle = \frac{|0\rangle + \alpha|1\rangle}{\sqrt{1+\alpha^2}} \otimes \frac{|0\rangle + \beta|1\rangle}{\sqrt{1+\beta^2}} = \frac{1}{\sqrt{(1+\alpha^2)(1+\beta^2)}}(|00\rangle + \beta|01\rangle + \alpha|10\rangle + \alpha\beta|11\rangle) = \frac{1}{\sqrt{(1+\alpha^2)(1+\beta^2)}} \begin{pmatrix} 1 \\ \beta \\ \alpha \\ \alpha\beta \end{pmatrix}$

$$F_4|\psi\rangle = \frac{1}{2\sqrt{(1+\alpha^2)(1+\beta^2)}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ \beta \\ \alpha \\ \alpha\beta \end{pmatrix} = \frac{1}{2\sqrt{(1+\alpha^2)(1+\beta^2)}} \begin{pmatrix} 1+\beta+\alpha+\alpha\beta \\ 1+i\beta-\alpha-i\alpha\beta \\ 1-\beta+\alpha-\alpha\beta \\ 1-i\beta-\alpha+i\alpha\beta \end{pmatrix}$$

$$= \frac{1}{2\sqrt{(1+\alpha^2)(1+\beta^2)}} \begin{pmatrix} (\alpha+1)(\beta+1) \\ (-\alpha+1)(i\beta+1) \\ (\alpha+1)(-\beta+1) \\ (-\alpha+1)(-i\beta+1) \end{pmatrix}$$

A state $|\psi\rangle$ in $(\mathbb{C}^2)^{\otimes 2}$ is entangled if it's not a product state (if it's not a product of two states $\varphi \otimes \phi$, $\varphi, \phi \in \mathbb{C}^2$)

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \varphi \otimes \phi = \begin{pmatrix} \varphi_1\phi_1 \\ \varphi_1\phi_2 \\ \varphi_2\phi_1 \\ \varphi_2\phi_2 \end{pmatrix}$$

For which values of α and β can $|\psi\rangle$ be written as a product state?

Only for $(\beta + 1) = (i\beta + 1)$ and $(-\beta + 1) = (-i\beta + 1)$ **or** $(\beta + 1) = (-\beta + 1)$ and $(i\beta + 1) = (-i\beta + 1)$.

And this is both only true for $\beta = 0$. For $\beta = 0$ the $|\psi\rangle$ is always a product state because then

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}\sqrt{\alpha^2+1}} \left((\alpha + 1)|0\rangle + (1 - \alpha)|1\rangle\right).$$

Thus, for $\beta \neq 0$, $|\psi\rangle$ is entangled.