Quantum Computing

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Homework 2

Exercise 2.7

Verifiy that $|w\rangle \equiv$ (1, 1) and $|v\rangle \equiv$ (1, -1) are orthogonal.

Two vectors are orthogonal, if their inner product is zero. (The inner product $\langle a|b\rangle$ is the projection from b to a.)

(1, 1), (1, -1)) = (1*, 1*) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ = 1 - 1 = 0

What are the normalized forms of these vectors?

$\frac{|w\rangle}{\||w\rangle\|} = \frac{(1,1)}{\sqrt{2}}$ and $\frac{|v\rangle}{\||v\rangle\|} = \frac{(1,-1)}{\sqrt{2}}$

Exercise 2.8

Prove that the Gram-Schmidt procedure produces an orthonormal basis $(|v_1\rangle, ..., |v_d\rangle)$ for V given some basis $(|w_1\rangle, ..., |w_d\rangle)$.

$|v_1\rangle \equiv \frac{|w_1\rangle}{\||w_1\rangle\|}$

$|v_{k+1}\rangle \equiv \frac{|w_{k+1}\rangle - \sum_{i=1}^{k}\langle v_i|w_{k+1}\rangle|v_i\rangle}{\||w_{k+1}\rangle - \sum_{i=1}^{k}\langle v_i|w_{k+1}\rangle|v_i\rangle\|}$ for $1 \leq k \leq d-1$

For an orthonormal basis $(|v_1\rangle, ..., |v_d\rangle)$ it is: $\langle v_i|v_j\rangle = \delta_{i,j}$. It's obvious, that the norm of every vector produced by the Gram-Schmidt-Procedure is 1. It remains to show that $\langle v_i|v_j\rangle = 0$ if i $\neq$ j, 1 $\leq$ i,j $\leq$ d . I do that by induction over d.

1. base case: d = 2

$\langle v_1|v_2\rangle = ((\frac{|w_1\rangle}{\||w_1\rangle\|}), (\frac{|w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle\|}))$; write $|w_1\rangle = \alpha|v_1\rangle$ with $\alpha = \||w_1\rangle\| \in \mathbb{R}$, $\langle v_1|w_2\rangle \in \mathbb{C}$, inner product linear in the second argument, cunjugate-linear in the first $\Rightarrow$

$\langle v_1|v_2\rangle = \frac{\langle \alpha v_1|w_2\rangle - \langle v_1|w_2\rangle\langle \alpha v_1|v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle\|} = \frac{\alpha^*\langle v_1|w_2\rangle - \alpha^*\langle v_1|w_2\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle\|} = 0$

2. induction step d $\mapsto$ d + 1

"produce" a new orthonormal base vector $|v_{d+1}\rangle \equiv \frac{|w_{d+1}\rangle - \sum_{i=1}^{d}\langle v_i|w_{d+1}\rangle|v_i\rangle}{\||w_{d+1}\rangle - \sum_{i=1}^{d}\langle v_i|w_{d+1}\rangle|v_i\rangle\|}$

for every i,j with $1 \leq i, j \leq d$ it is:

$\langle v_j|v_{d+1}\rangle = \frac{\langle v_j|w_{d+1}\rangle - \sum_{i=1}^{d}\langle v_i|w_{d+1}\rangle\langle v_j|v_i\rangle}{\||w_{d+1}\rangle - \sum_{i=1}^{d}\langle v_i|w_{d+1}\rangle|v_i\rangle\|}$

$\langle v_j|v_i\rangle = 0$ for i $\neq$ j and $\langle v_j|v_i\rangle = 1$ for i = j (induction hypothesis) $\Rightarrow$

$\langle v_j|v_{d+1}\rangle = \frac{\langle v_j|w_{d+1}\rangle - \langle v_j|w_{d+1}\rangle}{\| |w_{d+1}\rangle - \sum_{i=1}^{d}\langle v_i|w_{d+1}\rangle|v_i\rangle \|} = 0$

The number of (linearly independent) vectors in both bases is the same $\Rightarrow$ they span the same vector space V.

Exercise 2.9

Completeness relation in the same space V with the orthonormal basis $|0\rangle, |1\rangle$

$A = I_v A I_v = \langle 0|A|0\rangle|0\rangle\langle 0| + \langle 0|A|1\rangle|0\rangle\langle 1| + \langle 1|A|0\rangle|1\rangle\langle 0| + \langle 1|A|1\rangle|1\rangle\langle 1|$

$A = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow (\langle 0|, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix})\ |0\rangle\langle 0| + (\langle 0|, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix})$ $|0\rangle\langle 1|\ +\ (\langle 1|, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix})\ |1\rangle\langle 0|\ +\ (\langle 1|, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix})\ |1\rangle\langle 1|\ =$ $(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})\ |0\rangle\langle 0|\ +\ (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix})\ |0\rangle\langle 1|\ +\ (\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})\ |1\rangle\langle 0|$ $+\ (\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix})\ |1\rangle\langle 1| = |0\rangle\langle 0|$ - $|1\rangle\langle 1|$

$A = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow (\langle 0|, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix})\ |0\rangle\langle 0| + (\langle 0|, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix})$ $|0\rangle\langle 1| + (\langle 1|, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix})\ |1\rangle\langle 0| + (\langle 1|, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix})\ |1\rangle\langle 1| = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix})$ $|0\rangle\langle 0| + (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -i \\ 0 \end{pmatrix})\ |0\rangle\langle 1| + (\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix})\ |1\rangle\langle 0| + (\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 0 \end{pmatrix})$ $|1\rangle\langle 1| = i|1\rangle\langle 0|$ - $i|0\rangle\langle 1|$

$A = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1|\ +\ |1\rangle\langle 0|$

$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0|\ +\ |1\rangle\langle 1|$

Exercise 2.10

Given a vector $\vec{b}$ with respect to the basis $|v_i\rangle$

$\vec{b} = \alpha_1|v_1\rangle + ... + \alpha_n|v_n\rangle \equiv (\alpha_1 ... \alpha_n)^T$

Operator $|v_j\rangle\langle v_k|$, $(1 \leq j, k \leq n)$

$(|v_j\rangle\langle v_k|)(\vec{b}) = \langle v_k|\vec{b}\rangle|v_j\rangle$, $\langle v_k|\vec{b}\rangle = \alpha_k$ (because the basis is orthonormal) $\Rightarrow$ The operator scales the vector $|v_j\rangle$ with the factor α_k

The Matrix representation of this operator is a n×n matrix where the k-th column is the vector $|v_j\rangle$ and the rest is filled with 0s.

$$\begin{pmatrix} 0 & ... & v_{j,1} & 0 & ... & 0 \\ 0 & ... & v_{j,2} & 0 & ... & 0 \\ 0 & ... & ... & 0 & ... & 0 \\ 0 & ... & v_{j,n-1} & 0 & ... & 0 \\ 0 & ... & v_{j,n} & 0 & ... & 0 \end{pmatrix} \text{ as for every } \vec{b} = \begin{pmatrix} \alpha_1 \\ ... \\ \alpha_k \\ ... \\ \alpha_n \end{pmatrix}$$

($v_{j,g}$=g-th coordinate of $|v_j\rangle$):

$$\begin{pmatrix} 0 & ... & v_{j,1} & 0 & ... & 0 \\ 0 & ... & v_{j,2} & 0 & ... & 0 \\ 0 & ... & ... & 0 & ... & 0 \\ 0 & ... & v_{j,n-1} & 0 & ... & 0 \\ 0 & ... & v_{j,n} & 0 & ... & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ ... \\ \alpha_k \\ ... \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_k v_{j,1} \\ ... \\ \alpha_k v_{j,k} \\ ... \\ \alpha_k v_{j,n} \end{pmatrix} = \alpha_k \; |v_j\rangle$$

Exercise 2.13

It is $(AB)^\dagger = B^\dagger A^\dagger$ and we defined $|v\rangle^\dagger \equiv \langle v| \Rightarrow (|w\rangle\langle v|)^\dagger = \langle v|^\dagger |w\rangle^\dagger = |v\rangle\langle w|$

Exercise 2.16

Show that any projector P satisfies the equation $P^2 = P$
It is $\langle i|i\rangle = 1$ because $|i\rangle$ are vectors of the orthonormal basis for V

$P \equiv \sum_{i=1}^k |i\rangle\langle i| \Rightarrow PP = PIP = \sum_{i=1}^k |i\rangle\langle i|I|i\rangle\langle i| = \sum_{i=1}^k \langle i|I|i\rangle|i\rangle\langle i| = \sum_{i=1}^k \langle i|i\rangle|i\rangle\langle i|$
$= \sum_{i=1}^k |i\rangle\langle i| = P$

Exercise 2.17

Show that a normal matrix is Hermitian iff it has real eigenvalues.

" $\Rightarrow$ " A normal Matrix N is diagonalizable (spectral decomposition) with respect to a basis B.

$$\text{N} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \alpha_n \end{pmatrix} \text{ The matrix N is Hermitian } \Leftrightarrow N = N^\dagger$$

$$N^\dagger = \begin{pmatrix} \alpha_1^* & 0 & 0 & 0 \\ 0 & \alpha_2^* & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \alpha_n^* \end{pmatrix}$$

$\Rightarrow \alpha_i = \alpha_i^* \; \forall i : 1 \leq i \leq n \Rightarrow \alpha_i \in \mathbb{R}$. And the α_i are the eigenvalues of the matrix N (characteristic polynom: $(\alpha_1 - \lambda)(\alpha_2 - \lambda)...(\alpha_n - \lambda)) \Rightarrow \square$

" $\Leftarrow$ " analogous

Exercise 2.22

To prove this, I take a look at the matrices $(A - \lambda_i I)$ for the **real** eigenvalues λ_i. Without loss of generality (the eigenvalues and eigenvectors for a matrix A are always the same, regardless of the chosen basis) I choose the diagonalized matrix representation of A with respect to a certain basis B (Hermitian matrices can always be diagonalized), where the diagonal elements are the **real** eigenvalues.

To calculate the eigenvectors for a certain λ_i one have to solve the equation $(A - \lambda_i I)\vec{x} = 0$ which is:

$$\begin{pmatrix} \lambda_1 - \lambda_i & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_i & 0 & 0 & 0 & 0 \\ 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_i - \lambda_i & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_n - \lambda_i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ ... \\ ... \\ x_n \end{pmatrix} = 0$$

Define for $1 \leq j, k \leq n$: $\zeta_{j,k} = \upsilon_{jk} \in \mathbb{R}$ (υ_{jk} can have any value in $\mathbb{R}$) if $\lambda_k - \lambda_j = 0$, 0 else

Thus, for every eigenvalue λ_i, all the vectors $\begin{pmatrix} \zeta_{i,1} \\ \zeta_{i,2} \\ ... \\ ... \\ \zeta_{i,n} \end{pmatrix}$ are eigenvectors to the eigenvalues λ_i. For two different eigenvalues λ_a and λ_b it is:

$\zeta_{a,i} = \upsilon_{ai} \Leftrightarrow \zeta_{b,i} = 0$ as $[(\lambda_a \neq \lambda_b) \wedge (\lambda_i - \lambda_a = 0) \Rightarrow (\lambda_i - \lambda_b \neq 0)]$

It follows, that the inner product of the eigenvectors $|a\rangle, |b\rangle$ corresponding to the two **different** eigenvalues λ_a, λ_b is always 0 $\Rightarrow$ the eigenvectors are orthogonal.

$$(|a\rangle, |b\rangle) = (\zeta_{a,1}, \zeta_{a,1}, ..., \zeta_{a,1}) \begin{pmatrix} \zeta_{b,1} \\ \zeta_{b,2} \\ ... \\ ... \\ \zeta_{b,n} \end{pmatrix} = 0$$

Exercise 2.23

A projector P is normal and therefore diagonizable $\Rightarrow$ there is a basis B so that

$P = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \alpha_n \end{pmatrix}$ with respect to this basis.

We proved that $P^2 = P$ for all projectors $P \Rightarrow$

$$P^2 = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \alpha_n \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \alpha_n \end{pmatrix}$$

$\Rightarrow \alpha_i^2 = \alpha_i \forall i : 1 \leq i \leq n \Rightarrow \alpha_i \in \{0, 1\}$

Again, the eigenvalues of a diagonalized matrix A are the diagonal elements $\{\alpha_i\}$ as the characteristic polynom $((\alpha_1 - \lambda)(\alpha_2 - \lambda)...(\alpha_n - \lambda))$ (which is the determinant of $(A - \lambda I)$) has to be 0 to calculate the eigenvectors.

Exercise 2.24

Show that a positive operator is necessarily Hermitian.

A positive operator B is defined to be an operator such that for **any** vector $|v\rangle$, $(|v\rangle, B|v\rangle)$ is a **real**, non-negative number.

Any Operator A, any vector $\vec{v}$:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & ... & a_{2,n-1} & a_{2,n} \\ ... & ... & ... & ... & ... \\ a_{n-1,1} & a_{n-1,2} & ... & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & ... & a_{n,n-1} & a_{n-1,n} \end{pmatrix}, \ |v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_{n-1} \\ v_n \end{pmatrix}$$

$$(|v\rangle, A|v\rangle) =$$

$$(\overline{v}_1 \ \overline{v}_2 \ ... \ \overline{v}_{n-1} \ \overline{v}_n) \begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & ... & a_{2,n-1} & a_{2,n} \\ ... & ... & ... & ... & ... \\ a_{n-1,1} & a_{n-1,2} & ... & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & ... & a_{n,n-1} & a_{n-1,n} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_{n-1} \\ v_n \end{pmatrix} =$$

$$\begin{aligned} & a_{1,1}v_1\overline{v}_1 + a_{1,2}v_2\overline{v}_1 + ... + a_{1,n-1}v_{n-1}\overline{v}_1 + a_{1,n}v_n\overline{v}_1 + \\ & a_{2,1}v_1\overline{v}_2 + a_{2,2}v_2\overline{v}_2 + ... + a_{2,n-1}v_{n-1}\overline{v}_2 + a_{2,n}v_n\overline{v}_2 + \\ & ... + ... + ... + ... + ... + \\ & a_{n-1,1}v_1\overline{v}_{n-1} + a_{n-1,2}v_2\overline{v}_{n-1} + ... + a_{n-1,n-1}v_{n-1}\overline{v}_{n-1} + a_{n-1,n}v_n\overline{v}_{n-1} + \\ & a_{n,1}v_1\overline{v}_n + a_{n,2}v_2\overline{v}_n + ... + a_{n,n-1}v_{n-1}\overline{v}_n + a_{n-1,n}v_n\overline{v}_n \end{aligned}$$

With that, I show now, that for every operator A, which is not Hermitian, there is a vector $|v\rangle$ (there are two vectors $|v_1\rangle$ and $|v_2\rangle$ for Case 2) such that $(|v\rangle, A|v\rangle) \in \mathbb{C} - \mathbb{R}$ (either $(|v_1\rangle, A|v_1\rangle)$ or $(|v_2\rangle, A|v_2\rangle) \in \mathbb{C} - \mathbb{R}$ for Case 2). There are two possible cases:

Case 1: $a_{i,i} \in \mathbb{C} - \mathbb{R}$ for one (or more) i (not in $\mathbb{R} \Rightarrow$ A is not Hermitian)

$$\text{Choose e.g. } |v\rangle = \begin{pmatrix} 0 \\ ... \\ v_i = 1 \\ 0 \\ ... \\ 0 \end{pmatrix} \Rightarrow (|v\rangle, A|v\rangle) = a_{i,i} \in \mathbb{C}$$

Case 2: $\exists i, j : a_{i,j} \neq \overline{a}_{j,i}$ ($\Rightarrow$ A is not Hermitian)

$$\text{Choose e.g. } |v_1\rangle = \begin{pmatrix} 0 \\ ... \\ v_i = 1 \\ 0 \\ ... \\ v_j = i \\ 0 \\ ... \\ 0 \end{pmatrix} \Rightarrow (|v_1\rangle, A|v_1\rangle) = a_{i,i} + a_{j,j} + ia_{i,j} - ia_{j,i}$$

Since $a_{i,i} \in \mathbb{R}$ and $a_{j,j} \in \mathbb{R}$ (otherwise we would have applied Case 1) and for two complex numbers $a_x = a_{xr} + ia_{xi}$ and $a_y = a_{yr} + ia_{yi}$: $ia_x - ia_y = ia_{xr} - a_{xi} - ia_{yr} + a_{yi} \in \mathbb{R}$ iff $a_{xr} = a_{yr}$.

Choose e.g. $|v_2\rangle = \begin{pmatrix} 0 \\ ... \\ v_i = 1 \\ 0 \\ ... \\ v_j = 1 \\ 0 \\ ... \\ 0 \end{pmatrix} \Rightarrow (|v_2\rangle, A|v_2\rangle) = a_{i,i} + a_{j,j} + a_{i,j} + a_{j,i}$

Since $a_{i,i} \in \mathbb{R}$ and $a_{j,j} \in \mathbb{R}$ (otherwise we would have applied Case 1) and for two complex numbers $a_x = a_{xr} + ia_{xi}$ and $a_y = a_{yr} + ia_{yi}$: $a_x + a_y = a_{xr} + ia_{xi} + a_{yr} + ia_{yi} \in \mathbb{R}$ iff $a_{xi} = -a_{yi}$.

Thus, I showed that if A is not Hermitian, there is at least one vector $|\psi\rangle$ such that $\langle\psi|A\psi\rangle \in \mathbb{C} - \mathbb{R}$. This stands in contradiction to the definition of a positive operator. $\square$

Exercise 4.3

Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_z(\pi/4)$.

$R_z(\pi/4) = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$ and $T = e^{i\pi/8} \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$ The global phase is $e^{i\pi/8}$.

Exercise 4.6

One can show, that a rotation of a Bloch vector $\vec{\lambda}$ by an angle α in S^2 about the x/y/z axis is equivalent to applying the operator $R_{x/y/z}$ to $cos(\theta/2) + e^{i\varphi} sin(\theta/2))$, up to a global phase.

I will show that with the $R_z(\theta)$ operator:

$R_z(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = R_z(\alpha) = e^{-i\alpha/2} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$

Applied to $\begin{pmatrix} cos(\theta/2) \\ e^{i\varphi} sin(\theta/2) \end{pmatrix}$ this is $\begin{pmatrix} cos(\theta/2) \\ e^{i\alpha} e^{i\varphi} sin(\theta/2) \end{pmatrix} = \begin{pmatrix} cos(\theta/2) \\ e^{i\varphi + \alpha} sin(\theta/2) \end{pmatrix}$

$\Rightarrow$ As we only increase the angle φ this is equivalent to a rotation about the z axis. (One can see that easily in the Bloch sphere graphical representation.) One can show this analogous for the other two matrices, but the rotation is then applied (depending on the chosen $\hat{n}$) to both angles. When you compare this to the 3-dimensional rotations on the 3-dimensional Bloch vector you get the same results.

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -sin(\alpha) \\ 0 & sin(\alpha) & cos(\alpha) \end{pmatrix}, R_y(\alpha) = \begin{pmatrix} cos(\alpha) & 0 & -sin(\alpha) \\ 0 & 1 & 0 \\ sin(\alpha) & 0 & cos(\alpha) \end{pmatrix}$$

Exercise 4.7

$XYX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
$= -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -Y$ and $XX = I$

$XR_y(\theta)X = X(cos(\frac{\theta}{2})I - isin(\frac{\theta}{2})Y)X = cos(\frac{\theta}{2})XIX$ - $isin(\frac{\theta}{2})XYX = cos(\frac{\theta}{2})I$ $+ isin(\frac{\theta}{2})Y = \begin{pmatrix} cos(\frac{\theta}{2}) & 0 \\ 0 & cos(\frac{\theta}{2}) \end{pmatrix} + \begin{pmatrix} 0 & sin(\frac{\theta}{2}) \\ -sin(\frac{\theta}{2}) & 0 \end{pmatrix} = \begin{pmatrix} cos(\frac{\theta}{2}) & sin(\frac{\theta}{2}) \\ -sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix}$ $= R_y(-\theta)$ because $sin(-\theta) = -sin(\theta)$ (antisymmetric function) and $cos(-\theta) = cos(\theta)$ (symmetric function).

Exercise 4.8

1. We know that $R_{\hat{n}}(\theta)$ rotates a state represented by a Bloch vector $\vec{\lambda}$ by an angle θ about the $\hat{n}$ axis of the Bloch sphere. Thus we can express every operation, up to a global phase $e^{i\alpha}$, on a state of a qubit with $R_{\hat{n}}(\theta)$. $\Rightarrow$ An arbitary single qubit operator can be written in the form $A = e^{i\alpha}R_{\hat{n}}(\theta)$.

   Show that all these operators A are unitary, which is $AA^\dagger = I$

   We know that X, Y, Z are unitary and that $R_{\hat{n}}(\theta) = e^{(-i\theta\hat{n}\cdot\vec{\sigma}/2)}$ with $\vec{\sigma} = (X, Y, Z)$. $(X, Y, Z)^\dagger = (X, Y, Z)$

   Thus $AA^\dagger = e^{i\alpha}R_{\hat{n}}(\theta)(e^{i\alpha})^\dagger R_{\hat{n}}(\theta)^\dagger = e^{i\alpha}e^{-i\alpha}e^{(-i\theta\hat{n}\cdot\vec{\sigma}/2)}e^{(i\theta\hat{n}\cdot\vec{\sigma}/2)} = e^{(i\alpha - i\alpha)}$ $e^{((i\theta\hat{n}\cdot\vec{\sigma}/2)-(i\theta\hat{n}\cdot\vec{\sigma}/2))} = e^0 e^0 = 1$ □

2. $\alpha = \pi/2$, $\hat{n} = \frac{1}{\sqrt{2}}(1\ 0\ 1)$, $\theta = \pi$

3. $\alpha = \pi/4$, $\hat{n} = (0\ 0\ 1)$, $\theta = \pi/2$