Quantum Computing

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Homework 3

Exercise 2.39

1. Show that the function $(.\ ,\ .)$ on $L_v \times L_v$ defined by $(A,B) \equiv tr(A^\dagger B)$ is an inner product function.

   - $(.\ ,\ .)$ has to be linear in the second argument:

     $(A, \sum_i \lambda_i B_i) = \sum_i \lambda_i (A, B_i) \equiv tr(A^\dagger \sum_i \lambda_i B_i) = \sum_i \lambda_i tr(A^\dagger B_i)$

     $tr(A^\dagger \sum_i \lambda_i B_i) = tr(A^\dagger(\lambda_1 B_1 + \lambda_2 B_2 + ... + \lambda_n B_n)) = tr(\lambda_1 A^\dagger B_1 + \lambda_2 A^\dagger B_2 + ... + \lambda_n A^\dagger B_n) =^{\mathbf{1}} tr(\lambda_1 A^\dagger B_1) + tr(\lambda_2 A^\dagger B_2) + ... + tr(\lambda_n A^\dagger B_n)$ $=^{\mathbf{2}} \lambda_1 tr(A^\dagger B_1) + \lambda_2 tr(A^\dagger B_2) + ... + \lambda_n tr(A^\dagger B_n) = \sum_i \lambda_i tr(A^\dagger B_i)$

     1 $tr(A+B) = tr(A) + tr(B)$
     2 $tr(zA) = z\ tr(A)$ (z complex number, A arbitrary)

   - Show: $tr(A^\dagger B) = tr(B^\dagger A)^\dagger$ (* and † have the same effect on the diagonal of a matrix and hence on the trace)

     It is $tr(A)^\dagger = tr(A^\dagger)$ because $\overline{(a+b)} = (\overline{a} + \overline{b})$ for complex numbers , $(A^\dagger)^\dagger = A$ and $(BA)^\dagger = A^\dagger B^\dagger \Rightarrow$

     $tr(A^\dagger B) = tr((B^\dagger A)^\dagger) = tr(B^\dagger A)^\dagger$

   - $(A,A) \equiv tr(A^\dagger A) \geq 0$ with equality if and only if A $\equiv$ zero matrix.

     Let B $= A^\dagger A = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{21} & ... & \overline{a}_{n1} \\ \overline{a}_{12} & \overline{a}_{22} & ... & \overline{a}_{n2} \\ ... & ... & ... & ... \\ \overline{a}_{1n} & \overline{a}_{2n} & ... & \overline{a}_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$

     Every diagonal element b_{ii} (the addends of the trace of B) $= a_{1i}\overline{a}_{1i} + a_{2i}\overline{a}_{2i} + ... + a_{ni}\overline{a}_{ni} \geq 0$ with equality for $a_{ij} = 0\ \forall\ i,j$: $1 \leq i,j \leq n$ (because $\forall\ a \in \mathbb{C}$: $a\overline{a} \in \mathbb{R}^{\geq 0}$ with $a\overline{a} = 0$ if and only if $a = 0$) $\Rightarrow$

     $(A,A) \equiv tr(A^\dagger A) \geq 0$ with equality if and only if A $\equiv$ zero matrix.

2. If V has d dimensions show that L_v has dimension d^2.

   Let $\{|\nu_i\rangle\}$, $1 \leq i \leq d$ be an orthonormal basis of V. There are d orthonormal vectors in this base. Claim: $\{|\nu_i\rangle\langle\nu_j|\}$, $1 \leq i,j \leq d$ is a basis of L_v.

   Consider an operator $A : V \rightarrow V$ that maps a vector space V into itself. Assume any basis set $B_v = \{|\nu_i\rangle\}$. We can write the result of A operating on one of the basis vectors $|\nu_i\rangle$ as

$A|\nu_i\rangle = \sum_j A_{ji}|\nu_j\rangle$ multiplie from the right by $\langle\nu_i|$ and sum over i:

$\sum_i A|\nu_i\rangle\langle\nu_i| = \sum_{i,j} A_{ji}|\nu_j\rangle\langle\nu_i|$ A is linear $\Rightarrow$

$A\sum_i |\nu_i\rangle\langle\nu_i| = \sum_{i,j} A_{ji}|\nu_j\rangle\langle\nu_i|$ It is $\sum_i |\nu_i\rangle\langle\nu_i| = 1$ and thus:

$A = \sum_{i,j} A_{ji}|\nu_j\rangle\langle\nu_i|$ (Every operator is a linear combination of the basis).

The base is orthogonal:

$tr((|\nu_i\rangle\langle\nu_j|)^\dagger|\nu_k\rangle\langle\nu_l|) = tr(|\nu_j\rangle\langle\nu_i|\nu_k\rangle\langle\nu_l| = tr(|\nu_j\rangle\langle\nu_l|)\delta_{ik}$
Definition of the trace:

$tr(|\nu_j\rangle\langle\nu_l|)\delta_{ik} = \delta_{ik}\sum_n(|\nu_n\rangle\langle\nu_n|)(|\nu_j\rangle\langle\nu_l|) = \delta_{ik}\sum_n\langle\nu_n|(|\nu_j\rangle\langle\nu_l|)|\nu_n\rangle = \delta_{ik}\delta_{jl} \Rightarrow$ There are $n \cdot n$ (because the base $\{|\nu_i\rangle\}$ has dimension n) vectors in the orthogonal base $\Rightarrow$ the dimension is n^2.

Exercise 2.40

$[X, Y] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$
$= 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2iZ$
$[Y, Z] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$
$= 2i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2iX$
$[Z, X] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$= 2i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2iY$

Exercise 2.41

$\{X, Y\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$
$= 0$

$\{Y, Z\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} +$
$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0$
$\{Z, X\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} +$
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0$

$I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}$

$$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}$$
$$Y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}$$
$$Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}$$

Exercise 2.47

Suppose A and B are Hermitian. Show that i[A,B] is Hermitian.

Let $C = i[A, B] = i(AB - BA)$

$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, $(BA)_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$

$\Rightarrow c_{ij} = i\sum_{k=1}^{n}(a_{ik}b_{kj} - b_{ik}a_{kj})$

To show:

1. $c_{ii} \in \mathbb{R}$
   $\underline{c_{ii}} = i\sum_{k=1}^{n}(a_{ik}b_{ki} - b_{ik}a_{ki}) =^{\mathbf{1}} i\sum_{k=1}^{n}(a_{ik}b_{ki} - \overline{b}_{ki}\overline{a}_{ik}) =^{\mathbf{2}} \sum_{k=1}^{n} i(a_{ik}b_{ki} - \overline{a_{ik}b_{ki}})$

   Let $x + iy = z \in \mathbb{C}, x, y \in \mathbb{R} : i(z - \overline{z}) = i(x + iy - x + iy) = i(i2y) = -2y \in \mathbb{R} \Rightarrow$

   $i(a_{ik}b_{ki} - \overline{a_{ik}b_{ki}}) \in \mathbb{R} \Rightarrow \sum_k i(a_{ik}b_{ki} - \overline{a_{ik}b_{ki}}) \in \mathbb{R}$

2. $c_{ij} = \overline{c}_{ji}$
   $c_{ij} = i\sum_{k=1}^{n}(a_{ik}b_{kj} - b_{ik}a_{kj})$
   $c_{ji} = i\sum_{k=1}^{n}(a_{jk}b_{ki} - b_{jk}a_{ki}) = \sum_{k=1}^{n}(ia_{jk}b_{ki} - ib_{jk}a_{ki})$
   $\overline{c}_{ji} = \sum_{k=1}^{n}\overline{(ia_{jk}b_{ki} - ib_{jk}a_{ki})} =^{\mathbf{2}} \sum_{k=1}^{n}(-i\overline{a}_{jk}\overline{b}_{ki} + i\overline{b}_{jk}\overline{a}_{ki}) =^{\mathbf{1}} \sum_{k=1}^{n}(ib_{kj}a_{ik} - ia_{kj}b_{ik}) = i\sum_{k=1}^{n}(a_{ik}b_{kj} - b_{ik}a_{kj}) = c_{ij}$

1 A and B are Hermitian $\Rightarrow a_{ii}, b_{ii} \in \mathbb{R} \forall i : 1 \leq i \leq n$ and $a_{ij} = \overline{a}_{ji}$ and $b_{ij} = \overline{b}_{ji}$ $\forall i, j : 1 \leq i, j \leq n$
2 $\overline{ab} = \overline{a}\overline{b}\ \forall a, b \in \mathbb{C}$

Exercise 2.26

$|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

$|\psi\rangle \otimes |\psi\rangle) = 1/2(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$

$$\equiv 1/2(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}) =$$
$$1/2(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\otimes\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle) = \frac{1}{2\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle) =$

$$\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = (1/2\sqrt{2}, 1/2\sqrt{2}, 1/2\sqrt{2}, 1/2\sqrt{2}, 1/2\sqrt{2}, 1/2\sqrt{2}, 1/2\sqrt{2}, 1/2\sqrt{2})^T$$

Exercise 2.28

1. **Show that** $(A \otimes B)^* = A^* \otimes B^*$

   Kronecker product ($A_{m\times n}, B_{p\times q}$, matrix representation of $A \otimes B$ has nq columns and mp rows):

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ ... & ... & ... & ... \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{pmatrix}$$

$$(A \otimes B)^* = \begin{pmatrix} (a_{11}B)^* & (a_{12}B)^* & ... & (a_{1n}B)^* \\ (a_{21}B)^* & (a_{22}B)^* & ... & (a_{2n}B)^* \\ ... & ... & ... & ... \\ (a_{m1}B)^* & (a_{m2}B)^* & ... & (a_{mn}B)^* \end{pmatrix} =^{\mathbf{1}} \begin{pmatrix} a_{11}^*B^* & a_{12}^*B^* & ... & a_{1n}^*B^* \\ a_{21}^*B^* & a_{22}^*B^* & ... & a_{2n}^*B^* \\ ... & ... & ... & ... \\ a_{m1}^*B^* & a_{m2}^*B^* & ... & a_{mn}^*B^* \end{pmatrix} = A^* \otimes B^*$$

   1 $\overline{aB} = \overline{a}\overline{B}$ since $\overline{ab} = \overline{a}\overline{b}$ $\forall a, b \in \mathbb{C}$

2. **Show that** $(A \otimes B)^T = A^T \otimes B^T$

$$(A \otimes B)^T = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ ... & ... & ... & ... \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{pmatrix}^T =^{\mathbf{1}} \begin{pmatrix} a_{11}B^T & a_{21}B^T & ... & a_{m1}B^T \\ a_{12}B^T & a_{22}B^T & ... & a_{m2}B^T \\ ... & ... & ... & ... \\ a_{1n}B^T & a_{2n}B^T & ... & a_{nm}B^T \end{pmatrix} = A^T \otimes B^T$$

   1: Let C = $(A \otimes B)$. Then $C_{ij} = a_{\lceil i/p \rceil \lceil j/q \rceil} b_{(((i-1) mod p)+1)\ (((j-1) mod q)+1)}$
   It's easy to see that, if C gets transposed $C_{ij} \rightarrow C_{ji}$: $a_{\lceil i/p \rceil \lceil j/q \rceil} \rightarrow a_{\lceil j/q \rceil \lceil i/p \rceil}$ and $b_{(((i-1) mod p)+1)\ (((j-1) mod q)+1)} \rightarrow b_{(((j-1) mod q)+1)\ (((i-1) mod p)+1)}$
   $\Rightarrow$ every B in the matrix representation has to be transposed.

3. **Show that** $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

   $(A \otimes B)^\dagger = ((A \otimes B)^T)^* =^{\mathbf{1}} (A^T \otimes B^T)^* =^{\mathbf{2}} (A^T)^* \otimes (B^T)^* = A^\dagger \otimes B^\dagger$

   1: 2.28 (2.)
   2: 2.28 (1.)

Exercise 2.30

$(A \otimes B)^\dagger =^{\mathbf{1}} A^\dagger \otimes B^\dagger =^{\mathbf{2}} A \otimes B$

[1]: 2.28 (3.)
[2]: A and B are Hermitian $\Rightarrow A^\dagger = A$ and $B^\dagger = B$

Exercise 2.33

The Hadamard operator on one qubit:

$$H = \frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|] = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)$$

Show explicitily that the Hadamard transform on n qubits, $H^{\otimes n}$, may be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x,y}(-1)^{x\cdot y}|x\rangle\langle y|$$

First I claim that $|x\rangle\langle y| \otimes |i\rangle\langle j| = (|x\rangle \otimes |i\rangle)(\langle y| \otimes \langle j|)$ for all x, y and all combinations of $i, j \in \{0, 1\}$ and prove that for the case i=j=0 (all other cases are analogue)

Matrix representation of $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$|x\rangle\langle y| \otimes |0\rangle\langle 0| = \begin{pmatrix} x_1y_1 & x_1y_2 & ... & x_1y_n \\ x_2y_1 & x_2y_2 & ... & x_2y_n \\ ... & ... & ... & ... \\ x_ny_1 & x_ny_2 & ... & x_ny_n \end{pmatrix} \otimes |0\rangle\langle 0| = \begin{pmatrix} x_1y_1|0\rangle\langle 0| & x_1y_2|0\rangle\langle 0| & ... & x_1y_n|0\rangle\langle 0| \\ x_2y_1|0\rangle\langle 0| & x_2y_2|0\rangle\langle 0| & ... & x_2y_n|0\rangle\langle 0| \\ ... & ... & ... & ... \\ x_ny_1|0\rangle\langle 0| & x_ny_2|0\rangle\langle 0| & ... & x_ny_n|0\rangle\langle 0| \end{pmatrix}$$

$$= \begin{pmatrix} x_1y_1 & 0 & x_1y_2 & 0 & ... & x_1y_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2y_1 & 0 & x_2y_2 & 0 & ... & x_2y_n & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ x_ny_1 & 0 & x_ny_2 & 0 & ... & x_ny_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(|x\rangle \otimes |0\rangle)(\langle y| \otimes \langle 0|) = \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ 0 \\ ... \\ x_n \\ 0 \end{pmatrix} (y_1\ 0\ y_2\ 0\ ...\ y_n\ 0) = \begin{pmatrix} x_1y_1 & 0 & x_1y_2 & 0 & ... & x_1y_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2y_1 & 0 & x_2y_2 & 0 & ... & x_2y_n & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ x_ny_1 & 0 & x_ny_2 & 0 & ... & x_ny_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, I use an Induction proof on n:

Base case: n = 1:

$$H^{\otimes 1} = \frac{1}{\sqrt{2}}\sum_{x,y}(-1)^{x\cdot y}|x\rangle\langle y| \text{ with } x, y \in \{0, 1\} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)$$

Since $|x\rangle\langle y| \otimes |i\rangle\langle j| = \{|x0\rangle\langle y0|, |x1\rangle\langle y0|, |x0\rangle\langle y1|, |x1\rangle\langle y1|\}\ \forall x, y$ and for all combinations of $i, j \in \{0, 1\}$ where $|x0\rangle$ means $|x\rangle \otimes |0\rangle$ ($x, y \equiv |\alpha_1\alpha_2...\alpha_n\rangle = ((...(|\alpha_1\rangle \otimes |\alpha_2\rangle) \otimes ...) \otimes |\alpha_n\rangle)$ with $\alpha_i \in \{0, 1\}$ it is:

$$\frac{1}{\sqrt{2^n}}\sum_{x,y}(-1)^{x\cdot y}|x\rangle\langle y| \otimes \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) = \frac{1}{\sqrt{2^{n+1}}}\sum_{x\in\{|x0\rangle,|x1\rangle\},y}(-1)^{x\cdot y}|x\rangle\langle y| \otimes (|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)$$

...

Finally, I think that the formula is not provable, because it's not right. One can already see that on the matrix representation for $H^{\otimes 2}$:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Every element in the matrix is "created" by $|i\rangle\langle j|$ or - $|i\rangle\langle j|$ for every possible combination of i,j $\in \{00, 01, 10, 11\}$ (the final matrix is the sum of these matrices). For example the element in row 4, column 4 is "created" by $|11\rangle\langle 11| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This stands in contradiction to the given formula (there it would be - $|11\rangle\langle 11|$ since $(00, 01, 10, 11) \rightarrow (0, 1, 2, 3)$ and $3 * 3 = 9$ (odd)). The formula would be appropriate for something like $(\frac{1}{\sqrt{2}}(I+X))^{\otimes n} \otimes H$.

Exercise 2.54

Suppose A and B are commuting Hermitian operators. Prove that $exp(A)exp(B) = exp(A + B)$.

When A and B are commuting Hermitian operators, that is $[A, B] = AB - BA = 0$, then A and B are diagonal in the same orthonormal basis. Since A and B are Hermitian one can write:

$A = \sum_i \alpha_i |i\rangle\langle i|$ and $B = \sum_i \beta_i |i\rangle\langle i|$ with α_i and β_i are the **real** (A and B are Hermitian) eigenvalues and $|i\rangle$ is an orthonormal basis and each $|i\rangle$ is also an eigenvector of A / B with eigenvalue α_i / β_i (spectral decomposition).

Thus $exp(A)exp(B) = (\sum_i exp(\alpha_i)|i\rangle\langle i|)(\sum_i exp(\beta_i)|i\rangle\langle i|) = \sum_i exp(\alpha_i)exp(\beta_i)|i\rangle\langle i|$
$= \sum_i exp(\alpha_i + \beta_i)|i\rangle\langle i| = exp(\sum_i(\alpha_i + \beta_i)|i\rangle\langle i|) = exp((\sum_i \alpha_i|i\rangle\langle i|) + (\sum_i \beta_i|i\rangle\langle i|))$
$= exp(A + B)$.

Exercise 2.56

U is unitary and thus normal $\Rightarrow$ U can be written as U = $\sum_j \lambda_j |j\rangle\langle j|$, where λ_j are the eigenvalues of U, $|j\rangle$ is an orthonormal basis and each $|j\rangle$ an eigenvector of U with eigenvalue λ_j (spectral decomposition).

It is $UU^\dagger$ = I and thus $\sum_j \lambda_j \overline{\lambda}_j |j\rangle\langle j| = \sum_j |j\rangle\langle j| \Rightarrow$

$\lambda_j \overline{\lambda}_j = 1 \Rightarrow \lambda_j \in \{e^{i((2k)\pi+\varphi)})\}, 0 \leq \varphi \leq 2\pi$ because $e^{i((2k)\pi+\varphi)} e^{-i((2k)\pi+\varphi)} = e^0 = 1$

$log(e^{i((2k)\pi+\varphi)}) = i((2k)\pi + \varphi), k \in \mathbb{Z}$

Thus, $K \equiv -ilog(U) = \sum_j -i\alpha_j |j\rangle\langle j|$ with $\alpha_j = log(\lambda_j)$, $i((2k)\pi + \varphi) \in \mathbb{R}$ $\forall j$ because $-ii((2k)\pi + \varphi) = (2k)\pi + \varphi \in \mathbb{R} \Rightarrow$

K = $\sum_j \mu_j |j\rangle\langle j|$ with $\mu_j \in \mathbb{R} \Rightarrow$ K is Hermitian. (For a diagonal matrix A with $a_{ii} \in \mathbb{R}$ it is $A = A^\dagger$). $A = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \mu_n \end{pmatrix}$