Quantum Computing

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Homework 5

Exercise 1

Redo quantum teleportation with the Bell state $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$.

Qubit to beam: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Input state: $|\psi_0\rangle = (\alpha|0\rangle + \beta|1\rangle)(\frac{|01\rangle - |10\rangle}{\sqrt{2}}) = \frac{1}{\sqrt{2}}[\alpha|0\rangle(|01\rangle - |10\rangle) + \beta|1\rangle(|01\rangle + |10\rangle)]$.

The first two qubits belong to Alice, the third belongs to Bob.
Alice sends her qubits through a $CNOT$ gate:

$|\psi_1\rangle = \frac{1}{\sqrt{2}}[\alpha|0\rangle(|01\rangle - |10\rangle) + \beta|1\rangle(|11\rangle + |00\rangle)]$.

Then, Alice sends the first of her qubits through a Hadamard gate:

$|\psi_2\rangle = \frac{1}{2}[\alpha(|0\rangle + |1\rangle)(|01\rangle - |10\rangle) + \beta(|0\rangle - |1\rangle)(|11\rangle + |00\rangle)] =$

$\frac{1}{2}[\alpha|001\rangle - \alpha|010\rangle + \alpha|101\rangle - \alpha|110\rangle + \beta|011\rangle + \beta|000\rangle - \beta|111\rangle - \beta|100\rangle] =$

$\frac{1}{2}[|00\rangle(\alpha|1\rangle + \beta|0\rangle) + |01\rangle(\beta|1\rangle - \alpha|0\rangle) + |10\rangle(\alpha|1\rangle - \beta|0\rangle) + |11\rangle(-\alpha|0\rangle - \beta|1\rangle)]$.

To get the original state $|\psi_0\rangle$, Bob has to apply the following operations to the state he measured:

- If Alice has measured $|00\rangle$: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- If Alice has measured $|01\rangle$: $-Z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- If Alice has measured $|10\rangle$: $ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- If Alice has measured $|11\rangle$: $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Exercise 2

The average value for a measurement is: $\langle\psi|M|\psi\rangle$ where M is the observable.

Here $M = X_1 Z_2$, $|\psi\rangle = (|00\rangle + |11\rangle)\sqrt{2}$

X_1 applied to $|\psi\rangle \Rightarrow (|10\rangle + |01\rangle)\sqrt{2}$ (flips the first qubit)

Z_2 applied to $(|10\rangle + |01\rangle)\sqrt{2} \Rightarrow (|10\rangle - |01\rangle)\sqrt{2} = |\psi^{'}\rangle$
Average Value of the measurement is now $\langle\psi|\psi^{'}\rangle$

Kronecker representation:

$$\langle\psi|\psi^{'}\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

Exercise 3

1. Show that $(Q \otimes R)(S \otimes T) = QS \otimes RT$
Suppose $|v\rangle$ and $|w\rangle$ arbitrary (state) vectors.

$((Q \otimes R)(S \otimes T))(|v\rangle \otimes |w\rangle) = (Q \otimes R)(S|v\rangle \otimes T|w\rangle) = QS|v\rangle \otimes RT|w\rangle = (QS \otimes RT)(|v\rangle \otimes |w\rangle)$

2. Show that $QQ = I$, $RR = I$, $SS = I$ and $TT = I$.

For Q, other cases are analogue:

$$Q = \begin{pmatrix} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{pmatrix}. \quad QQ = \begin{pmatrix} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{pmatrix}\begin{pmatrix} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{pmatrix}$$

$= \begin{pmatrix} q_1^2 + q_2^2 + q_3^2 & 0 \\ 0 & q_1^2 + q_2^2 + q_3^2 \end{pmatrix} = \mathrm{I}$ (because $\vec{q}$ is a unit vector in three dimensions, i.e. $\sqrt{q_1^2 + q_2^2 + q_3^2} = 1$).

$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 =$

$$\begin{aligned}
&(Q \otimes S)(Q \otimes S) + (Q \otimes S)(R \otimes S) + (Q \otimes S)(R \otimes T) - (Q \otimes S)(Q \otimes T) + \\
&(R \otimes S)(Q \otimes S) + (R \otimes S)(R \otimes S) + (R \otimes S)(R \otimes T) - (R \otimes S)(Q \otimes T) + \\
&(R \otimes T)(Q \otimes S) + (R \otimes T)(R \otimes S) + (R \otimes T)(R \otimes T) - (R \otimes T)(Q \otimes T) - \\
&(Q \otimes T)(Q \otimes S) - (Q \otimes T)(R \otimes S) - (Q \otimes T)(R \otimes T) + (Q \otimes T)(Q \otimes T) =^{\mathbf{1.}}
\end{aligned}$$

$$\begin{aligned}
&Q^2 \otimes S^2 + QR \otimes S^2 + QR \otimes ST - Q^2 \otimes ST + \\
&RQ \otimes S^2 + R^2 \otimes S^2 + R^2 \otimes ST - RQ \otimes ST + \\
&RQ \otimes TS + R^2 \otimes TS + R^2 \otimes T^2 - RQ \otimes T^2 - \\
&Q^2 \otimes TS - QR \otimes TS - QR \otimes T^2 + Q^2 \otimes T^2 =^{\mathbf{2.}}
\end{aligned}$$

$4I + QR \otimes ST - RQ \otimes ST + RQ \otimes TS - QR \otimes TS = 4I + (QR - RQ) \otimes (ST - TS)$
$= 4I + [Q, R] \otimes [S, T]$

$\langle A\rangle = \langle\psi|A|\psi\rangle = E(A)$ for observables A and arbitrary state vectors $|\psi\rangle$.

It is $\langle I\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1$ and $\langle[Q,R]\rangle \leq \langle QR\rangle + \langle RQ\rangle \leq^{(Cauchy-Schwarz)} 2\langle Q\rangle\langle R\rangle$.

And $\langle Q\rangle \leq 1$, $\langle R\rangle \leq 1$, as the eigenvalues of Q, R are ± 1 (proven in hw4; $E(Q) = \langle Q\rangle = p_q(1) - p_q(-1)) \leq 1$ and $E(R) = \langle R\rangle = p_r(1) - p_r(-1)) \leq 1$ (For[S, T] analogue)

The upper bound for $\langle W^2\rangle = \langle(4I + [Q,R]\otimes[S,T])\rangle$ is $4 + 4 = 8$.

Thus $\langle Q\otimes S\rangle + \langle R\otimes S\rangle + \langle R\otimes T\rangle - \langle Q\otimes T\rangle \leq \sqrt{\langle W^2\rangle} \leq 2\sqrt{2}$.

Exercise 4.1

$|\psi\rangle = \sum_{I=0}^{2^n-1} a_I|I\rangle$.

$E(\psi) = \frac{4}{n}\sum_{j=1}^{n} Area^2(\psi_j^0, \psi_j^1)$ with $Area^2(v,w) = |v|^2|w|^2 sin^2\theta = |v|^2|w|^2 - |\langle v,w\rangle|^2$.

$|\psi_j^0\rangle = \sum_{\{I|j^{th}bit=0\}} a_I|\widehat{I}_j\rangle$ and $|\psi_j^1\rangle = \sum_{\{I|j^{th}bit=1\}} a_I|\widehat{I}_j\rangle$.

(a) $E(\frac{|000\rangle+|111\rangle}{\sqrt{2}})$, n = 3, for every j: $Area^2(\psi_j^0,\psi_j^1) = Area^2(\frac{|00\rangle}{\sqrt{2}}, \frac{|11\rangle}{\sqrt{2}}) =$

$$\frac{1}{2}\frac{1}{2} - |(\frac{1}{\sqrt{2}}\ 0\ 0\ 0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}|^2 = \frac{1}{4} - 0. \text{ Thus } E(\psi) = \frac{4}{3}\sum_{j=1}^{3}\frac{1}{4} = \frac{4}{3}\frac{3}{4} = 1$$

(b) $E(\frac{\sum_{I=0}^{7}|I\rangle}{\sqrt{8}})$, n = 3, for every j: $Area^2(\psi_j^0,\psi_j^1) = Area^2(\frac{\sum_{I=0}^{3}|I\rangle}{\sqrt{8}}, \frac{\sum_{I=0}^{3}|I\rangle}{\sqrt{8}})$

$$= Area^2(\begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix}) = \frac{1}{2}\frac{1}{2} - |(\frac{1}{\sqrt{8}}\ \ \frac{1}{\sqrt{8}}\ \ \frac{1}{\sqrt{8}}\ \ \frac{1}{\sqrt{8}})(\begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix}|^2 =$$

$\frac{1}{4} - \frac{1}{4} = 0$. Thus $E(\psi) = \frac{4}{3}\sum_{j=1}^{3} 0 = 0$

Exercise 4.2

Since $Area^2(\psi_j^0,\psi_j^1) = |\psi_j^0|^2|\psi_j^1|^2 sin^2\theta$, $Area^2(\psi_j^0,\psi_j^1) \leq |\psi_j^0|^2|\psi_j^1|^2$ (because $sin^2\theta \leq 1$).

With $|\psi\rangle = \sum_{I=0}^{2^n-1} a_I|I\rangle$
it is $|\psi_j^0|^2 + |\psi_j^1|^2 = a_0^2 + a_1^2 + ... + a_{2^n-1}^2 = 1$.

Assume $|\psi_j^0|^2 = \frac{n+h}{2n}$ and $|\psi_j^1|^2 = \frac{n-h}{2n}$, $h \in \mathbb{R}$. $(|\psi_j^0|^2 + |\psi_j^1|^2 = 1)$

$\frac{n+h}{2n}\frac{n-h}{2n} = \frac{n^2-h^2}{4n^2}$ gets max for $h = 0 \Rightarrow |\psi_j^0|^2|\psi_j^1|^2 \leq \frac{1}{4}$.

$\Rightarrow E(\psi) = \frac{4}{n}\sum_{j=1}^{n} Area^2(\psi_j^0,\psi_j^1) \leq \frac{4}{n}\sum_{j=1}^{n}\frac{1}{4} = \frac{4}{n}\frac{n}{4} = 1$.

As $Area^2(\psi_j^0,\psi_j^1) = \sum_{i<j}|v_i w_j - v_j w_i|^2 \geq 0$, $E(\psi) \geq 0$.

Exercise 4.3

"$\Rightarrow$":

$E(\psi) = 0$ iff for every j: $Area^2(\psi_j^0, \psi_j^1) = 0$. This is only the case, if ψ_j^0 and ψ_j^1 are lineary dependent, i.e. $\psi_j^0 = \lambda_j \psi_j^1$ for **all** $1 \leq j \leq n$, $\lambda_j \in \mathbb{C}$.

$\psi = |0\rangle \otimes \psi_1^0 + |1\rangle \otimes \psi_1^1 = (|0\rangle + \lambda_1 |1\rangle) \otimes \psi_1^0 = (U \otimes I)(|0\rangle \otimes \chi)$ where U is an unitary operator (like in homework 2, e.g. a rotation on the bloch sphere) and $\chi \in \mathbb{C}^{\otimes(n-1)}$.

Then: $0 = E(\psi) = E(|0\rangle \otimes \chi) = 0 + \sum_{j=2}^{n} Area^2(|0\rangle \otimes \chi_j^0, |0\rangle \otimes \chi_j^1) = E(\chi)$.
(Because $||0\rangle \otimes \psi|^2 = |\begin{pmatrix} \psi \\ 0 \end{pmatrix}|^2 = |\psi|^2$).

By Induction, ψ is a product state.

"$\Leftarrow$":

If ψ is a product state, ψ_j^0 is parallel to ψ_j^1 and $E(\psi) = 0$.