The Implication Problem For Disjunctive Rules

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Basic Definitions & Examples

Definition 1.1 *Let* S *be a finite set, let* $X \subseteq S$*, and let* $\mathcal{Y}$ *a set of subsets of* S*. A* disjunctive rule *over* S *is an expression of the form* $X \rightarrow \mathcal{Y}$*. The number of subsets of* S *in* $\mathcal{Y}$*, denoted* $|\mathcal{Y}|$*, is called the* order *of* $X \rightarrow \mathcal{Y}$*.*

Example 1.2 *Let* $S = \{A, B, C\}$*. The following are disjunctive rules of order respectively 0, 1, 2, and 3:*
$\{A, B, C\} \rightarrow \emptyset$*;*
$\{C\} \rightarrow \{\{A, B\}\}$*;*
$\{A, B\} \rightarrow \{\{A\}, \{A, B\}\}$*; and*
$\emptyset \rightarrow \{\{A, B\}, \{B, C\}, \{A, C\}\}$*.*

Basic Definitions & Examples

Triviality	**Strong Transitivity**
$$\frac{}{X \cup Y \rightarrow \mathcal{Y} \cup \{Y\}}$$	$$\frac{X \rightarrow \mathcal{Y} \qquad \forall Y \in \mathcal{Y} : Y \cup W \rightarrow \mathcal{Z}}{X \cup W \rightarrow \mathcal{Z}}$$

Figure 1: Inference system for disjunctive rules.

Basic Definitions & Examples

Definition 1.3 *Let* S *be a finite set, let* $\mathcal{C}$ *be a set of disjunctive rules over* S*, and let* c *be a disjunctive rule over* S*. We say that* c *can be* derived *from* $\mathcal{C}$ *if there exists a finite sequence of disjunctive rules* $c_1, \ldots, c_n$ *such that*

1. *for* $i = 1, \ldots, n$*,* c_i *is either given (i.e., an element of* $\mathcal{C}$*) or* c_i *can be derived from some of the constraints* $c_1, \ldots, c_{i-1}$ *using an inference rule;*
2. $c_n = c$*.*

The sequence $c_1, \ldots, c_n$ *is called a* derivation.

Basic Definitions & Examples

Definition 1.4 *Let* S *be a finite set, let* $\mathcal{C}$ *be a set of disjunctive rules over* S*, and let* c *be a disjunctive rule over* S*. Furthermore, let* u *be the highest order of a disjunctive rule in* $\mathcal{C} \cup \{c\}$ *and let* ℓ *be the lowest order of a disjunctive rule in* $\mathcal{C} \cup \{c\}$*.*

If all the disjunctive rules in a derivation of c *from* $\mathcal{C}$ *have order at most* u*, we say that the derivation is* upper bounded*. If all the disjunctive rules in a derivation of* c *from* $\mathcal{C}$ *have order at least* ℓ*, we say that the derivation is* lower bounded*. Finally, if a derivation is both* lower bounded *and* upper bounded*, we say that it is* bounded.

Basic Definitions & Examples

Example 1.5 *Let* $S = \{A, B, C, D\}$, *let* $\mathcal{C} = \{\{A\} \rightarrow \{\{B, C\}, \{C, D\}\}, \{C\} \rightarrow \{\{D\}\}\}$, *and let* c *be* $\{A, B\} \rightarrow \{\{D\}\}$. *Observe that all disjunctive rules in* $\mathcal{C} \cup \{c\}$ *are of order 1 or 2. Let's look at a bounded derivation:*

(1)	$\{C\}$	$\rightarrow$	$\{\{D\}\}$	(given)
(2)	$\{B, C, D\}$	$\rightarrow$	$\{\{D\}\}$	(**triviality**)
(3)	$\{B, C\}$	$\rightarrow$	$\{\{D\}\}$	(**s.t.** $W = \{B, C$
(4)	$\{A\}$	$\rightarrow$	$\{\{B, C\}, \{C, D\}\}$	(given)
(5)	$\{A, B\}$	$\rightarrow$	$\{\{D\}\}$	(**s.t.** $W = \{B\}$)

Basic Definitions & Examples

Augmentation

$$\frac{X \rightarrow \mathcal{Y}}{X \cup W \rightarrow \mathcal{Y}}$$

Figure 2: Additional inference rule derivable from the inference system in Figure 1.

Basic Definitions & Examples

Definition 1.6 *Let S be a finite set, let $X \subseteq S$, and let $\mathcal{Y}$ be a set of subsets of S. The* lattice decomposition *of X relative to $\mathcal{Y}$, is defined by*

$$L(X, \mathcal{Y}) = [X, S] - \left(\bigcup_{Y \in \mathcal{Y}} [Y, S] \right).$$

Definition 1.7 *Let S be a finite set and let $\mathcal{Y}$ be a set of subsets of S. A subset W of S is called a* witness *of $\mathcal{Y}$ if $W \subseteq \bigcup \mathcal{Y}$, and for each $Y \in \mathcal{Y}$, $Y \cap W \neq \emptyset$. A subset W of S is called a* minimal witness *of $\mathcal{Y}$ if it is a witness of $\mathcal{Y}$ and does not properly contain another witness of $\mathcal{Y}$.*

Basic Definitions & Examples

We denote by $\mathcal{W}^+(\mathcal{Y})$ the set of all witnesses of $\mathcal{Y}$ and by $\mathcal{W}^-(\mathcal{Y})$ the set of all minimal witnesses of $\mathcal{Y}$. We can now rewrite lattice decompositions as follows

Proposition 1.8 *Let S be a finite set, let $X \subseteq S$, and let $\mathcal{Y}$ be a set of subsets of S. Then*

$$L(X, \mathcal{Y}) = \bigcup_{W \in \mathcal{W}^+(\mathcal{Y})} [X, \overline{W}] = \bigcup_{W \in \mathcal{W}^-(\mathcal{Y})} [X, \overline{W}].$$

Basic Definitions & Examples

Theorem 1.9 *Let S be a finite set, let $X \subseteq S$, and let $\mathcal{Y}$ be a set of subsets of S. Let $\mathcal{C}$ be a set disjunctive rules over S and let c be the disjunctive rule $X \rightarrow \mathcal{Y}$. Then $\mathcal{C} \vdash c$ if and only if $L(X, \mathcal{Y}) \subseteq \bigcup_{X' \rightarrow \mathcal{Y}' \in \mathcal{C}} L(X', \mathcal{Y}')$.*

A careful inspection of the proof of the above theorem reveals that we can make a stronger statement on the syntactic nature of the derivations involved.

Theorem 1.10 *Let S be a finite set, let $\mathcal{C}$ be a set of disjunctive rules over S, and let c be a disjunctive rule over S. If $\mathcal{C} \vdash c$ then there exists a bounded derivation of $\mathcal{C} \vdash c$.*

Differentials and Densities

Definition 1.11 *Let S be a finite set, let $F : 2^S \longrightarrow \mathbf{R}$, and let let $\mathcal{Y}$ be a set of subsets of S. The $\mathcal{Y}$*-differential *of F is the function $\Delta^{\mathcal{Y}} F : 2^S \longrightarrow \mathbf{R}$ defined recursively:*
$\Delta^{\emptyset} F(X) = F(X)$ *and*
$\Delta^{\mathcal{Y} \cup \{Y\}} F(X) = \Delta^{\mathcal{Y}} F(X) - \Delta^{\mathcal{Y}} F(X \cup Y)$
for each $X \subseteq S$.

The analogy with the definition of derivatives of real functions is obvious. Notice that $|\mathcal{Y}|$ corresponds to the order of differentiation.

Differentials and Densities

Definition 1.12 *Let S be a finite set, and let $F : 2^S \to \mathbf{R}$. The* density *of F is the function $\Delta F : 2^S \to \mathrm{R}$ defined by $\Delta F(X) = \Delta^{\mathcal{A}(\overline{X})} F(X)$, for each $X \subseteq S$.*

The following relationship between a function and its density justifies the name:

Proposition 1.13 *Let S be a finite set and let $F : 2^S \to \mathbf{R}$. Then, for each $X \subseteq S$, $F(X) = \sum_{X \subseteq U \subseteq S} \Delta F(U)$.*

The analogy with densities in probability theory is obvious.

Differentials and Densities

Additionally, it turns out that the differential of a set-valued function can also be expressed in terms of its densities.

Proposition 1.14 *Let S be a finite set and let $F : 2^S \to \mathbf{R}$. Then, for each $X \subseteq S$, $\Delta^{\mathcal{Y}} F(X) = \sum_{U \in L(X,\mathcal{Y})} \Delta F(U)$.*

Differentials as Constraints

We can interpret disjunctive rules as constraints on set-valued functions into the reals.

Definition 1.15 *Let S be a finite set, let X be a subset of S, and let $\mathcal{Y}$ be a set of subsets of S. Let $F : 2^S \to \mathbf{R}$. We say that F* satisfies *the disjunctive rule $X \to \mathcal{Y}$ if $\Delta^{\mathcal{Y}} F(X) = 0$.*

Differentials as Constraints

Relative to this notion of satisfaction, we define the logical implication problem for disjunctive rules given a class of set-valued functions $\mathcal{F}$.

Definition 1.16 *Let S be a finite set, let $\mathcal{C}$ be a set of disjunctive rules over S, let c be a disjunctive rule over S, and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. We say that $\mathcal{C}$* logically implies *c* under *$\mathcal{F}$, and write $\mathcal{C} \models_{\mathcal{F}} c$, if each function $F \in \mathcal{F}$ that satisfies all disjunctive rules in $\mathcal{C}$ also satisfies the disjunctive rule c.*

Soundness

Definition 2.17 (Soundness property) *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. Let $\ell, u \in \mathbf{N}$ such that $0 \leq \ell \leq u \leq 2^{|S|}$. We say that $\mathcal{F}$ has the $[\ell, u]$*-soundness property *if, for each set $\mathcal{C}$ of disjunctive rules over S of order at least ℓ and at most u, and each disjunctive rule c over S of order at least ℓ and at most u, we have that $\mathcal{C} \vdash c$ implies $\mathcal{C} \models_{\mathcal{F}} c$.*

Soundness

Definition 2.18 (Augmentation and zero-density p.)

- *We say that $\mathcal{F}$ has the* $[\ell, u]$-augmentation property *if, for each function F in $\mathcal{F}$ and for each disjunctive rule $X \rightarrow \mathcal{Y}$ over S of order at least ℓ and at most u, then whenever F satisfies $X \rightarrow \mathcal{Y}$, we have that F also satisfies $X \cup W \rightarrow \mathcal{Y}$, for each $W \subseteq S$.*
- *We say that $\mathcal{F}$ has the* $[\ell, u]$-zero-density property *if, for each function F in $\mathcal{F}$ and for each disjunctive rule $X \rightarrow \mathcal{Y}$ over S of order at least ℓ and at most u, then whenever F satisfies $X \rightarrow \mathcal{Y}$, we have that $\Delta F(U) = 0$, for all $U \in L(X, \mathcal{Y})$.*

Soundness

Each of these properties is equivalent to the $[\ell, u]$-soundness property:

Theorem 2.19 *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. Let $\ell, u \in \mathbf{N}$ such that $0 \leq \ell \leq u \leq 2^{|S|}$. Then the following statement are equivalent:*

1. *$\mathcal{F}$ has the $[\ell, u]$-augmentation property;*
2. *$\mathcal{F}$ has the $[\ell, u]$-zero-density property; and*
3. *$\mathcal{F}$ has the $[\ell, u]$-soundness property.*

Completeness

First, we define the notion of completeness formally.

Definition 2.20 (Completeness property) *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. Let $\ell, u \in \mathbf{N}$ such that $0 \leq \ell \leq u \leq 2^{|S|}$. We say that $\mathcal{F}$ has the* $[\ell, u]$-completeness property *if, for each set $\mathcal{C}$ of disjunctive rules over S of order at least ℓ and at most u, and each disjunctive rule c over S of order at least ℓ and at most u, we have that $\mathcal{C} \models_{\mathcal{F}} c$ implies $\mathcal{C} \vdash c$.*

Completeness

We now present two properties that guarantee $[0, 2^{|S|}]$-completeness and $[1, 2^{|S|}]$- completeness, respectively. To do so, we need to introduce the following notions:

Definition 2.21 *Let S be a finite set and let $V \subseteq S$.*

- *We define the* Kronecker density function *of V, denoted δ_V, as the function from 2^S into the reals for which $\delta_V(V) = 1$ and $\delta_V(X) = 0$ if $X \neq V$.*
- *We define the* Kronecker-induced function *of V, denoted F_V, as the function whose density function is the* Kronecker density function *of V, i.e., $F_V(X) = \sum_{X \subseteq U \subseteq S} \delta_V(U)$, for each $X \subseteq S$.*

Completeness

Based on these notions, we can define the following properties of classes of set-valued functions into the reals:

Definition 2.22 (Strong and weak Kronecker properties) *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued function from 2^S into the reals.*

- *We say that $\mathcal{F}$ has the* strong Kronecker property *if, for each subset V of S, there exists a real number $c_V \neq 0$ such that the function $c_V F_V$ is in $\mathcal{F}$.*
- *We say that $\mathcal{F}$ has the* weak Kronecker property *if, for each* strict *subset V of S, there exist real numbers $c_V \neq 0$ and d_V such that the function $c_V F_V + d_V$ is in $\mathcal{F}$.*

Completeness

Theorem 2.23 *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals.*

- *If $\mathcal{F}$ has the strong Kronecker property it also has the $[0, 2^{|S|}]$-completeness property.*
- *If $\mathcal{F}$ has the weak Kronecker property it also has the $[1, 2^{|S|}]$-completeness property.*

Soundness and Completeness

Definition 2.24 *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. Let $\ell, u \in \mathbf{N}$ such that $0 \leq \ell \leq u \leq 2^{|S|}$. We say that $\mathcal{F}$ has the $[\ell, u]$-sound-and-completeness property if $\mathcal{F}$ has both, the $[\ell, u]$-soundness and $[\ell, u]$-completeness property.*

Soundness and Completeness

Theorem 2.25 *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. Let $\ell, u \in \mathbf{N}$ such that $0 \leq \ell \leq u \leq 2^{|S|}$.*

- *If $\mathcal{F}$ has the $[\ell, u]$-augmentation property (respectively, the $[\ell, u]$-zero-density property) and the strong Kronecker property, then $\mathcal{F}$ has also the $[\ell, u]$-sound-and-completeness property.*
- *In the above statement, the strong Kronecker property may be replaced by the weak Kronecker property, provided $\ell \geq 1$.*

Soundness and Completeness

Ideally, we would like to *characterize* $[\ell, u]$-sound-and-completeness. For $\ell = 0$ and $\ell = 1$, we actually can.

Lemma 2.26 *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. Let $u \in \mathbf{N}$, $1 \leq u \leq 2^{|S|}$.*

- *If $\mathcal{F}$ has the $[0, u]$-sound-and-completeness property, it also has the strong Kronecker property.*
- *If $\mathcal{F}$ has the $[1, u]$-sound-and-completeness property, it also has the weak Kronecker property.*

Soundness and Completeness

Theorem 2.27 *Let S be a finite set and let $\mathcal{F}$ be a class of set-valued functions from 2^S into the reals. Let $u \in \mathbf{N}$, $1 \leq u \leq 2^{|S|}$. Then,*

- *$\mathcal{F}$ has the $[0, u]$-sound-and-completeness property if and only if it has the $[0, u]$-augmentation property (respectively, the $[0, u]$-zero-density property) and the strong Kronecker property; and*
- *$\mathcal{F}$ has the $[1, u]$-sound-and-completeness property if and only if it has the $[1, u]$-augmentation property (respectively, the $[1, u]$-zero-density property) and the weak Kronecker property.*

Applications

Example 3.28 (All functions) *Let S be a finite non-empty set. In The set of* all *set-valued functions does not have the $[0, 2^{|S|}]$-soundness property. However, since this class has the strong Kronecker property it does have the $[0, 2^{|S|}]$-completeness property.*

Applications

Example 3.29 (Frequent itemset mining) *Given a finite set S of items and a list $\mathcal{B}$ of subsets of S. The* support *of an itemset $X \subseteq S$ is defined as $S^{\mathcal{B}}(X) = |\mathcal{B}(X)|$ where $\mathcal{B}(X) = \{i \mid X \subseteq \mathcal{B}[i]\}$. The class of all support functions over S has the $[0, 2^{|S|}]$-augmentation property, whence the $[0, 2^{|S|}]$-soundness property. One can also easily verify that the class of all support functions has the strong Kronecker property. Thus, this class has the $[0, 2^{|S|}]$-sound-and-completeness property.*

Applications

Definition 3.30 (Choquet capacities) *Let S be a finite set, let F be a function from 2^S into the reals, and let k be a natural number, $k \geq 1$.*

- *The function F is a* positive k-alternating capacity *if, for each subsets X of S and for each non-empty set $\mathcal{Y}$ of at most k subsets of S, $\Delta^{\mathcal{Y}} F(X) \geq 0$.*
- *The function F is a* negative k-alternating capacity *if, for each subset X of S and for each non-empty set $\mathcal{Y}$ of at most k subsets of S, $\Delta^{\mathcal{Y}} F(X) \leq 0$.*

Applications

Corollary 3.31 *Let S be a finite set, and let $k \geq 1$. Every class of positive (respectively, negative) $(k+1)$-alternating capacities over S has the $[1, k]$-sound-and-completeness property if and only if it has the weak Kronecker property.*

Theorem 3.32 *Let S be a finite set, and let $k \geq 1$. Every class of positive $(k+1)$-alternating capacities over S with $k \geq 1$ has the $[0, k]$-soundness property. Moreover, it has the $[0, k]$-sound-and-completeness property if and only if it has the strong Kronecker property.*

Applications

Now, these results follow directly:

- The class of Shannon entropy functions has the $[1, 1]$-sound-and-completeness property. Compare this case to functional dependencies!
- The class of all plausibility functions over S has the $[1, 2^{|S|}]$-sound-and-completeness property;
- The class of all probability functions over S has the $[1, 2^{|S|}]$-soundness property but not the weak Kronecker property;
- The class of all $(k+1)$-alternating capacities, $k \geq 1$, has the $[1, k]$-sound-and-completeness property.