On Implication Problems for Disjunctive Constraints

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Disjunctive Statements as Syntactic Patterns

Observation

In many problem domains of computer science data is “restricted” by constraints that can be specified syntactically as disjunctive statements of the form $X \to \{Y_1, \ldots, Y_n\}$, with $X, Y_1, \ldots, Y_n$ pairwise disjoint subsets of some finite set S. (We will call n the order of the disjunctive statements.)

Database Systems

- Relational database with schema S; relation instance $\mathbf{r}$
- $\mathbf{r}$ satisfies functional dependency $X \rightarrow Y$ if, for any tuples t_1 and t_2 in $\mathbf{r}$, $t_1[X] = t_2[X]$ implies $t_1[Y] = t_2[Y]$
- $\mathbf{r}$ satisfies multivalued dependency $X \twoheadrightarrow Y|Z$, with $X \cup Y \cup Z = S$, if $\mathbf{r}$ can be decomposed losslessly into its projections on $X \cup Y$ and $X \cup Z$
- Used to removes redundancy and to prevent update anomalies, leading to a better database design
- FDs and MVDs can be re-written as disjunctive statements of order 1 ($X \rightarrow \{Y\}$) and 2 ($X \rightarrow \{Y, Z\}$), respectively.

Database Systems – Example

$\mathbf{r} :=$

a	b	c	d
a1	b1	c1	d1
a1	b1	c1	d2
a1	b2	c2	d1
a2	b1	c3	d1

- $S = \{a, b, c, d\}$ schema
- **r** satisfies the functional dependency $\{a, b\} \rightarrow \{c\}$
- Corresponds to the disjunctive statement $\{a, b\} \rightarrow \{\{c\}\}$

Database Systems – Implication Problem

$$\frac{\begin{array}{c} X_1 \rightarrow Y_1 \\ X_2 \rightarrow Y_2 \\ \ldots \\ X_k \rightarrow Y_k \end{array}}{X \rightarrow Y \ (??)}$$

Implication Problem for Functional Dependencies

Database Systems – Inference Systems

Reflexivity

$$\frac{}{X \supseteq Y : X \rightarrow Y}$$

Transitivity

$$\frac{X \rightarrow Y \quad Y \rightarrow Z}{X \rightarrow Z}$$

Augmentation

$$\frac{X \rightarrow Y}{X \cup W \rightarrow Y}$$

Union

$$\frac{X \rightarrow Y \quad X \rightarrow Z}{X \rightarrow Y \cup Z}$$

Figure: Sound and complete inference system for functional dependencies.

Database Systems – Implication Problem cont.

$$\frac{\begin{array}{c} X_1 \rightarrow Y_1 \mid Z_1 \\ X_2 \rightarrow Y_2 \mid Z_2 \\ \dots \\ X_k \rightarrow Y_k \mid Z_n \end{array}}{X \rightarrow Y \mid Z \text{ (??)}}$$

Implication Prob. for Multivalued Dependencies

Database Systems – Inference Systems cont.

Triviality

$$\frac{}{X \rightarrow \emptyset \mid \overline{X}}$$

Transitivity

$$\frac{\begin{array}{c} X \rightarrow Y \mid W \\ Y \rightarrow Z \mid V \end{array}}{X \rightarrow Z - Y \mid W \cup (Y \cap V)}$$

Augmentation

$$\frac{X \rightarrow Y \cup W \mid Z}{X \cup W \rightarrow Y \mid Z}$$

Symmetry

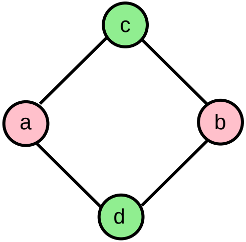
$$\frac{X \rightarrow Y \mid Z}{X \rightarrow Z \mid Y}$$

Figure: Sound and complete inference system for multivalued dependencies.

Reasoning under Uncertainty

- S finite set of random variables, P discrete probability measure
- P satisfies the *conditional independence statement* $I(Y, Z|X)$, with X, Y, and Z pairwise disjoint subsets of S, if for every assignment $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ to the variables in X, Y, and Z, $P(\mathbf{x})P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = P(\mathbf{x}, \mathbf{y})P(\mathbf{x}, \mathbf{z})$
- Central concept for graphical models (permits factorization of multivariate probability distributions)
- Conditional independence statements can be re-written as disjunctive statements of order 2 ($X \rightarrow \{Y, Z\}$).

Reasoning under Uncertainty – Example



- $S = \{a, b, c, d\}$
- Different separation criteria in graphical models “encode” conditional independence statements
- Markov network above *represents* CI statements $I(a, b|\{c, d\})$ and $I(c, d|\{a, b\})$

Reasoning under Uncertainty – Inference System

Triviality

$$\frac{}{I(\emptyset, \overline{X} | X)}$$

Contraction

$$\frac{\begin{array}{c} I(Y, Z | X) \\ I(Y, W | X \cup Z) \end{array}}{I(Y, Z \cup W | X)}$$

Weak Augmentation

$$\frac{I(Y, Z \cup W | X)}{I(Y, Z | X \cup W)}$$

Decomposition

$$\frac{I(Y, Z \cup W | X)}{I(Y, Z | X)}$$

Figure: Sound, but not complete, inference system for conditional independence statements; “the semi-graphoid axioms”

Propositional Logic

- S set of *propositional variables*, $\mathbf{w}$ *truth assignment* over S
- $\mathbf{w}$ satisfies an *implication formula* $X \rightarrow Y$, with X and Y disjoint subsets of S, if and only if $\mathbf{w}$ satisfies the propositional formula $\bigwedge_{x \in X} x \rightarrow \bigvee_{y \in Y} y$
- Notice that each propositional formula can be rewritten into an equivalent conjunction of implication formulae
- An implication formula $X \rightarrow Y$ can be re-written as the disjunctive statement $X \rightarrow \{\{y\} \mid y \in Y\}$ of order $|Y|$.

Propositional Logic – Example

- $S = \{p, q, r, s\}$
- $(p \vee q \vee \neg r \vee \neg s) \wedge (p \vee r) \wedge (\neg q \vee \neg s)$ equivalent to $(r \wedge s \rightarrow p \vee q) \wedge (\textbf{true} \rightarrow p \vee r) \wedge (q \wedge s \rightarrow \textbf{false})$
- Corresponds to the disjunctive statements $\{r, s\} \rightarrow \{\{p\}, \{q\}\}$, $\emptyset \rightarrow \{\{p\}, \{r\}\}$, and $\{q, s\} \rightarrow \emptyset$

Propositional Logic – Inference System

Augmentation

$$\frac{X \rightarrow Y}{X \cup W \rightarrow Y}$$

Strong Resolution

$$\frac{\begin{array}{c} X \rightarrow Y \cup Z \\ \forall z \in Z : X \cup \{z\} \rightarrow Y \cup V \end{array}}{X \rightarrow Y \cup V}$$

Composition

$$\frac{\begin{array}{c} X \rightarrow Y \\ X \rightarrow Z \end{array}}{X \rightarrow Y \cup Z}$$

Decomposition

$$\frac{X \rightarrow Y \cup Z}{X \rightarrow Y}$$

Figure: Sound and complete inference system for implication formulae.

Cooperative Game Theory

- Multi-player games in which players can form coalitions for joint cooperations
- S non-empty set of players, *worth function* w assigns to each subset X of S a value $w(X)$ representing the combined worth of the coalition X in a game
- w satisfies the interaction statement $A(X, i, j)$ if player i and j act without interference when joining the coalition X in a game
- Interaction statements can be re-written as disjunctive statements of order 2 ($X \rightarrow \{\{i\}, \{j\}\}$).

Cooperative Game Theory – Inference System

Triviality

$$\frac{}{A(X, \overline{X}, \emptyset)}$$

Contraction

$$\frac{\begin{array}{c} A(X, Z, Y) \\ A(X \cup Z, W, Y) \end{array}}{A(X, Z \cup W, Y)}$$

Weak Augmentation

$$\frac{A(X, Z \cup W, Y)}{A(X \cup W, Z, Y)}$$

Decomposition

$$\frac{A(X, Z \cup W, Y)}{A(X, Z, Y)}$$

Figure: Sound inference system for interaction statements.

Disjunctive Statements as Additive Constraints

These examples are “syntactically uniform.” Even the inference systems are somewhat similar. But what about the semantics?

Observation

All of the previously mentioned implication problems can be expressed as implication problems for additive constraints on specific classes of real-valued functions.

Database Systems – FDs and MVDs

Given a relation schema S and a relation instance r over S. Let P be the uniform probability distribution over the tuples of r, i.e., for each tuple t in r, $P(t) = 1/|r|$. The *Shannon entropy* H_s is defined by

$$H_r(X) = -\sum_{x \in \pi_X(s)} P^X(x) \log(P^X(x)).$$

- r satisfies the functional dependency $X \rightarrow Y$ if and only if $H_r(X) - H_r(X \cup Y) = 0$
- r satisfies the multivalued dependency $X \twoheadrightarrow Y|Z$ if and only if $H_r(X) - H_r(X \cup Y) - H_r(X \cup Z) + H_r(X \cup Y \cup Z) = 0$

Reasoning under Uncertainty – CI statements

Given a set S of random variables and a probability measure P over S. Let H_P be the relative entropy (Kullback-Leibler divergence). The *multi-information function* $M_P : 2^S \rightarrow [0, \infty)$ is defined by $M_P(\emptyset) = 0$ and $M_P(X) = H_P(P^X | \prod_{x \in X} P^{\{x\}})$, for each non-empty subset X of S.

- P satisfies $I(Y, Z|X)$ if and only if $M_P(X) - M_P(X \cup Y) - M_P(X \cup Z) + M_P(X \cup Y \cup Z) = 0$

Propositional Logic

Given a set S of propositional variables and a truth-assignment $\mathbf{w}$ over S, we define the real-valued function $W_{\mathbf{w}} : 2^S \rightarrow \mathbf{R}$ by $W_{\mathbf{w}}(X) = 1$ if each variable in X evaluates to **true** under $\mathbf{w}$, and $W_{\mathbf{w}}(X) = 0$ otherwise.

- $\mathbf{w}$ satisfies the implication formula $X \rightarrow Y$, if and only if $\sum_{\mathcal{Z} \subseteq \mathcal{Y}} (-1)^{|\mathcal{Z}|} W_{\mathbf{w}}(X \cup \bigcup_{Z \in \mathcal{Z}} Z) = 0$, where $\mathcal{Y} = \{\{y\} \mid y \in Y\}$
- **Example:** $S = \{p, q, r\}$. $\mathbf{w}$ satisfies $\{p\} \rightarrow \{q, r\}$ if and only if $W_{\mathbf{w}}(\{p\}) - W_{\mathbf{w}}(\{p, q\}) - W_{\mathbf{w}}(\{p, r\}) + W_{\mathbf{w}}(\{p, q, r\}) = 0$.

Cooperative Game Theory

Let S be a non-empty set of players and $w : 2^S \rightarrow \mathbf{R}$ be a worth function of a specific cooperative game.

- The interaction statement $A(X, i, j)$ holds in the game (S, w) if and only if $w(X \cup \{i, j\}) - w(X \cup \{i\}) - w(X \cup \{j\}) + w(X) = 0$

Summary Pt. 1

Numerous semantics (FDs, MVDs, CI, ...)

Specific class of real-valued functions $\mathcal{F}$ (implicit bounds on order; here: $[2,2]$)

$$\frac{\begin{array}{c} X_1 \to \{Y_1, Z_1\} \\ X_2 \to \{Y_2, Z_2\} \\ \cdots \\ X_k \to \{Y_k, Z_k\} \end{array}}{X \to \{Y, Z\}\ (??)} \quad \Leftrightarrow \quad \frac{\begin{array}{c} F(X_1) + F(X_1 \cup Y_1 \cup Z_1) = F(X_1 \cup Y_1) + F(X_1 \cup Z_1) \\ F(X_2) + F(X_2 \cup Y_2 \cup Z_2) = F(X_2 \cup Y_2) + F(X_2 \cup Z_2) \\ \cdots \\ F(X_k) + F(X_k \cup Y_k \cup Z_k) = F(X_k \cup Y_k) + F(X_k \cup Z_k) \end{array}}{F(X) + F(X \cup Y \cup Z) = F(X \cup Y) + F(X \cup Z)\ (??)}$$

- Given an implication problem (e.g., for conditional independence statements), we can reduce it to the corresponding equivalent implication problem on additive constraints
- For every semantics (e.g., conditional independence) we get a different class of real-valued functions $\mathcal{F}$ (e.g., class of multi-information functions induced by probability measures)

Summary Pt. 1

The Big Question: Can we relate certain properties of $\mathcal{F}$ to properties of the corresponding implication problem for additive constraints, namely, existence and structure of sound and/or complete inference systems, non-existence of finite axiomatizations, computational complexity of the implication problem, etc.?

All these findings would be applicable to implication problems that can be reduced to this framework (including, of course, the ones we have seen earlier).

Differentials

Definition (Differentials)

Let S be a finite set, let F be a real-valued function over S, and let $\mathcal{Y}$ be a set of subsets of S. The $\mathcal{Y}$-*differential* of F is the real-valued function over S recursively defined by $\Delta^{\emptyset}F(X) = F(X)$ and, for each $X \subseteq S$, $\Delta^{\mathcal{Y}\cup\{Y\}}F(X) = \Delta^{\mathcal{Y}}F(X) - \Delta^{\mathcal{Y}}F(X \cup Y)$. The size of $\mathcal{Y}$ is called the *order* of the differential.

Example

Important special cases are the first- and second-order differentials. According to Definition 1, we have, for a finite set S, a real-valued function F over S, and $X, Y, Z \subseteq S$, that
$\Delta^{\{Y\}}F(X) = F(X) - F(X \cup Y)$ and
$\Delta^{\{Y,Z\}}F(X) = F(X) - F(X \cup Y) - F(X \cup Z) + F(X \cup Y \cup Z)$.

Differentials and Additive Constraints

Definition (Additive Constraint)

Let S be a finite set, let $X \subseteq S$ and $\mathcal{Y} = \{Y_1, \dots Y_n\} \subseteq 2^S$ such that $\{X\} \cup \mathcal{Y}$ consists of pairwise disjoint sets, and let F be a real-valued function over S. Then, F satisfies the *disjunctive statement* $X \rightarrow \mathcal{Y}$ if and only if $\Delta^{\mathcal{Y}} F(X) = 0$.

Example

Let $S = \{a, b, c\}$ and F be a real-valued function with $F(\{a, b, c\}) = F(\{a, b\}) = F(\{a, c\}) = F(\{a\})$. Then, F satisfies $\{a\} \rightarrow \{\{b\}, \{c\}\}$ because $\Delta^{\{\{b\},\{c\}\}} F(\{a\}) =$ $F(\{a, b, c\}) - F(\{a, b\}) - F(\{a, c\}) + F(\{a\}) = 0$.

Bounded Implication Problem for Additive Constraints

Definition (Bounded Implication Problem)

For a finite set S, a set of disjunctive statements $\mathcal{C}$ over S of order at least ℓ and at most u, a disjunctive statement c over S of order at least ℓ and at most u, and a class of real-valued functions $\mathcal{F}_S$ over S, we say that $\mathcal{C}$ *logically implies* c relative to $\mathcal{F}_S$, and write $\mathcal{C} \models_{\mathcal{F}_S} c$, if every function $F \in \mathcal{F}_S$ that satisfies all disjunctive statements in $\mathcal{C}$ also satisfies c.

The *implication problem* is deciding logical implication with S as a parameter of the problem. This means that, for each finite set S, an appropriate set of real-valued functions over S is given.

Bounded Implication Problem for Additive Constraints

Example

$S = \{a, b, c\}$ and let $\mathcal{F}$ be the class of all increasing functions (i.e., $F(A) - F(AB) \leq 0$ for all S and all $F \in \mathcal{F}$). Let's look at an instance of the $[1, 1]$-bounded implication problem:

$$\frac{\{a\} \rightarrow \{\{b, c\}\}}{\{a, b\} \rightarrow \{\{c\}\}\ (??)} \quad \Leftrightarrow \quad \frac{F(\{a\}) - F(\{a, b, c\}) = 0}{F(\{a, b\}) - F(\{a, b, c\}) = 0\ (??)}$$

Yes, because $\forall F \in \mathcal{F}$:
$0 \geq F(\{a, b\}) - F(\{a, b, c\}) \geq F(\{a\}) - F(\{a, b, c\}) = 0.$

Augmentation

$$\frac{X \rightarrow \mathcal{Y}}{X \cup W \rightarrow \mathcal{Y}}$$

Strong Transitivity

$$\frac{\begin{array}{c} X \rightarrow \mathcal{Y} \cup \mathcal{Z} \\ \forall Z \in \mathcal{Z} : X \cup Z \rightarrow \mathcal{Y} \cup \mathcal{V} \end{array}}{X \rightarrow \mathcal{Y} \cup \mathcal{V}}$$

Composition

$$\frac{\begin{array}{c} X \rightarrow \mathcal{Y} \cup \{Y\} \\ X \rightarrow \mathcal{Y} \cup \{Z\} \end{array}}{X \rightarrow \mathcal{Y} \cup \{Y \cup Z\}}$$

Decomposition

$$\frac{X \rightarrow \mathcal{Y} \cup \{Y \cup Z\}}{X \rightarrow \mathcal{Y} \cup \{Y\}}$$

Triviality

$$\frac{}{X \rightarrow \mathcal{Y} \cup \{\emptyset\}}$$

Figure: Inference system template $\mathcal{K}$.

Triviality

$$\frac{}{X \rightarrow \mathcal{Y} \cup \{\emptyset\}}$$

Weak Augmentation

$$\frac{X \rightarrow \mathcal{Y} \cup \{Y \cup Z\}}{X \cup Z \rightarrow \mathcal{Y} \cup \{Y\}}$$

Weak Contraction

$$\frac{X \rightarrow \mathcal{Y} \cup \{Y\} \cup \{Z \cup W\} \quad X \cup Y \rightarrow \mathcal{Y} \cup \{Z\} \cup \{W\}}{X \rightarrow \mathcal{Y} \cup \{Y \cup Z\} \cup \{W\}}$$

Weak Addition

$$\frac{X \rightarrow \mathcal{Y} \cup \{Y \cup Z\}}{X \rightarrow \mathcal{Y} \cup \{Y\} \cup \{Z\}}$$

Figure: Semi-graphoid inference system template $\mathcal{G}$.

Soundness

Definition

We say that an inference system $\mathcal{I}$ is $[\ell, u]$-*sound* relative to $\mathcal{F}$ if, for each finite set S, for each set $\mathcal{C}$ of disjunctive statements over S, and for each single disjunctive statement c over S, all of order at least ℓ and at most u, we have that $\mathcal{C} \vdash_{\mathcal{I}} c$ implies $\mathcal{C} \models_{\mathcal{F}_S} c$.

Lemma

If $\mathcal{K}$ is $[\ell, u]$-sound relative to $\mathcal{F}$, then $\mathcal{G}$ is $[\ell, u]$-sound relative to $\mathcal{F}$.

Soundness of $\mathcal{K}$

Theorem

Let $\ell \leq u$ be bounds and let $\mathcal{F}$ be a class of real-valued functions. Then the following statements are equivalent:

(1) *Augmentation and decomposition are $[\ell, u]$-sound relative to $\mathcal{F}$;*

(2) *$\mathcal{F}$ has the $[\ell, u]$-zero-density property; and*

(3) *$\mathcal{K}$ is $[\ell, u]$-sound relative to $\mathcal{F}$.*

Soundness of $\mathcal{K}$ – Examples

Example

By Theorem 9 we have that $\mathcal{K}$ and $\mathcal{G}$ are $[1, 1]$-sound for the implication problem of functional dependencies since augmentation and decomposition are sound for functional dependencies. In addition, $\mathcal{K}$ and $\mathcal{G}$ are $[1, u]$-sound for implication formulae in propositional logic, for any upper bound u, because augmentation and decomposition are $[1, u]$-sound.

Soundness of $\mathcal{G}$

Theorem

Let $\ell \leq u$ be bounds and let $\mathcal{F}$ be a class of real-valued functions. Then the following statements are equivalent:

1. *Weak augmentation is $[\ell, u]$-sound relative to $\mathcal{F}$ for saturated disjunctive statements;*
2. *$\mathcal{F}$ has the $[\ell, u]$-zero-density property for saturated disjunctive statements;*
3. *$\mathcal{K}$ is $[\ell, u]$-sound relative to $\mathcal{F}$ for saturated disjunctive statements; and*
4. *$\mathcal{G}$ is $[\ell, u]$-sound relative to $\mathcal{F}$ for saturated disjunctive statements.*

Soundness of $\mathcal{G}$ – Examples

Example

By Theorem 11, $\mathcal{G}$ and inference system $\mathcal{K}$ are $[2, 2]$-sound for both the implication problem for multivalued dependencies and the implication problem for saturated conditional independence statements since weak augmentation is $[2, 2]$-sound in both cases.

Completeness

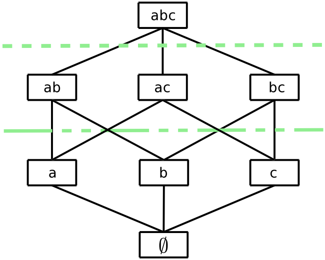
Definition

We say that $\mathcal{I}$ is $[\ell, u]$-*complete* relative to $\mathcal{F}$ if, for each finite set S, for each set $\mathcal{C}$ of disjunctive statements over S, and for each single disjunctive statement c over S, all of order at least ℓ and at most u, we have that $\mathcal{C} \models_{\mathcal{F}_S} c$ implies $\mathcal{C} \vdash_{\mathcal{I}} c$.

Lemma

If $\mathcal{G}$ is $[\ell, u]$-complete relative to $\mathcal{F}$, then $\mathcal{K}$ is $[\ell, u]$-complete relative to $\mathcal{F}$.

Completeness – Kronecker Property on $\mathcal{S}^i(S)$



- “Kronecker property on $\mathcal{S}^i(S)$” is a property of a class of real-valued functions with respect to a “region” of the lattice corresponding to a finite set S
- Above lattice illustrates $\mathcal{S}^0(S) = 2^S$, $\mathcal{S}^1(S)$, and $\mathcal{S}^2(S)$ for $S = \{a, b, c\}$

Completeness

Theorem

Let $\ell \leq u$ be bounds and let $\mathcal{F}$ be a class of real-valued functions. If, for every finite set S, $\mathcal{F}_S$ has the Kronecker property on $\mathcal{S}^\ell(S)$, then $\mathcal{K}$ is $[\ell, u]$-complete relative to $\mathcal{F}$.

Example

By Theorem 15, we have that $\mathcal{K}$ is $[1, 1]$-complete for the implication problem for functional dependencies since the class of Shannon entropy functions over a relation schema S has the Kronecker property on $\mathcal{S}^1(S)$. In addition, the inference system $\mathcal{K}$ is $[1, u]$-complete for implication formulae in propositional logic.

Completeness

Proposition

Let $\ell \leq u$ be bounds and let $\mathcal{F}$ be a class of real-valued functions.

1. *If $\ell \geq 1$ and $\mathcal{K}$ is $[\ell, u]$-sound and $[\ell, u]$-complete relative to $\mathcal{F}$, then, for each finite set S, $\mathcal{F}_S$ has the Kronecker property on $\mathcal{S}^\ell(S)$.*
2. *If $\ell \geq 2$ and $\mathcal{K}$ is $[\ell, u]$-sound and $[\ell, u]$-complete relative to $\mathcal{F}$ for saturated disjunctive statements, then $\mathcal{F}$ has the Kronecker property on $\mathcal{S}^\ell(S)$.*
3. *If $\ell \geq 2$ and $\mathcal{G}$ is $[\ell, u]$-sound and $[\ell, u]$-complete relative to $\mathcal{F}$ for saturated disjunctive statements, then $\mathcal{F}$ has the Kronecker property on $\mathcal{S}^\ell(S)$.*

Completeness

Theorem

Let $2 \leq \ell \leq u$ *be bounds and let* $\mathcal{F}$ *be a class of real-valued functions. If* $\mathcal{K}$ *is* $[\ell, u]$*-sound and* $[\ell, u]$*-complete relative to* $\mathcal{F}$ *for saturated disjunctive statements, then* $\mathcal{K}$ *is* $[\ell, u]$*-complete relative to* $\mathcal{F}$*.*

Completeness – Example

Example

Inference system $\mathcal{G}$ is $[2,2]$-complete for both the implication problem for multivalued dependencies and the implication problem for saturated conditional independence statements. $\mathcal{K}$ is also $[2,2]$-complete for these implication problems for saturated statements. By Example 12, $\mathcal{K}$ is also $[2,2]$-sound for these implication problems for saturated statements. Using Theorem 18, we may thus conclude that $\mathcal{K}$ is $[2,2]$-complete for both the implication problem for embedded multivalued dependencies and unrestricted conditional independence statements.

Dichotomy of Complexity

For many implication problems inference system $\mathcal{K}$ is both sound and complete.

Theorem

Let $1 \leq \ell \leq u$ *be bounds and let* $\mathcal{F}$ *be a class of real-valued functions, and let* $\mathcal{K}$ *be both* $[\ell, u]$*-sound and* $[\ell, u]$*-complete relative to* $\mathcal{F}$*. The corresponding implication problem is decidable in polynomial time if* $u = 1$*, and* coNP*-complete if* $u > 1$*.*

Example

The implication problem for implication formulae in propositional logic is coNP-complete. The implication problem for functional dependencies is polynomial time decidable.

Non-Existence of Axiomatization

Consider the following inference rule, Λ^n ($n \geq 3$):

$$\begin{array}{lcl} A_1 & \longrightarrow & \{A_2\} \cup \mathcal{Y} \\ A_2 & \longrightarrow & \{A_3\} \cup \mathcal{Y} \\ & \vdots & \\ A_{n-1} & \longrightarrow & \{A_n\} \cup \mathcal{Y} \\ A_n & \longrightarrow & \{A_1\} \cup \mathcal{Y} \\ \hline A_1 & \longrightarrow & \{A_n\} \cup \mathcal{Y} \end{array}$$

Non-Existence of Axiomatization

Theorem

Let $\ell \leq u$ be bounds and let $\mathcal{F}$ be a class of real-valued functions. If all of the following statements hold, then there does not exist a finite, complete axiomatization for the implication problem for additive constraints on $\mathcal{F}$ for disjunctive statements of order at least ℓ and at most u:

1. *Inference rule Λ^n is $[\ell, u]$-sound relative to $\mathcal{F}$ for every $n \geq 3$;*
2. *$\mathcal{K}$ is $[\ell, u]$-complete relative to $\mathcal{F}$; and*
3. *for each finite set S, $\mathcal{F}_S$ has the dual Kronecker property.*

Non-Existence of Axiomatization

- There exists no finite, complete axiomatization for the implication problem for CI statements relative to the class of discrete probability measures
- There exists no finite, complete axiomatization for the implication problem for CI statements relative to the class of **binary** discrete probability measures (was an open problem)
- There exists no finite, complete axiomatization for the implication problem for interaction statements relative to the class of all supermodular (submodular) functions in game theory (was an open problem)

Summary

- Many implication problems in computer science can be reduced to implication problems on additive constraints
- The only "parameters" of this implication problem are the class of real-valued functions $\mathcal{F}$ and the bounds on the disjunctive statements under consideration
- Properties of $\mathcal{F}$ can be related to properties of the implication problem (i.e., soundness of specific inference systems, implication complexity, existence of finite axiomatization)
- Provides insight into commonalities of different implication problems in computer science **and** can be used to solve open problems

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